INVARIANT CONTROL SETS ON FIBRE BUNDLES

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CONTENTS

To

Anita and Chica

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DECLARATION

The main results in this thesis are original. Theorems 1.2 and 2.4 appeared in a more specific form in [33]. Propositions 2.9 and E2.10.1 were published in [34] and Theorem 1.3 is to appear in [35].

Summary

This thesis deals with invariant control sets on fibre bundles. Although the context is nonlinear control theory, problems relevant to stochastic analysis are treated. It is divided in seven paragraphs or sections. §0. contains the basic notions used throughout. §1. is to prove a theorem on the accessibility property of control systems which is needed subsequently. §§2 and 3 deals with invariant control sets on fibre bundles. A method to handle them is developed. §§4,5 and 6 gives reality to the method of §§2 and 3 by analyzing invariant control sets on homogeneous spaces. §4. includes also a theorem on the controllability of semi-groups on semi-simple Lie groups.

Introduction

The notion of invariant control set of a control system was introduced by L. Arnold and W. Kliemann in connection with the study of diffusion processes on manifolds by means of nonlinear control theory (see Arnold and Kliemann [1] and Kliemann [21]. The definition of invariant control set is given below in §0).

The basic idea of the method used by them is to consider in association to the (Stratonovich) stochastic differential equation

$$dx = X(x)dt + \sum_{j=1}^{m} Y_j(x) \circ dW_j \tag{1}$$

the control system

$$\dot{x} = X(x) + \sum_{j=1}^{m} u_j \, Y_j(x) \tag{2}$$

which is obtained from (1) after a formal replacement of dW_j by u_j .

The link between (1) and (2) is provided by the support theorems of Stroock, Varadhan and Kunita. With the aid of these theorems, the supports of various probability measures related to the diffusion generated by (1) are characterized in terms of the control system (2).

Using this procedure, the concept of invariant control set which pertains to control theory, emerges when the supports of invariant measures of stochastic systems are considered. This way, in [1] the problem of unique ergodicity of systems like (1) is divided into the control problem of searching the invariant control sets and the question of deciding about the number of invariant measures inside an invariant control set.

In this thesis I am concerned with the investigation of invariant control sets of control systems. The context here is control theory and no reference to stochastics is made.

However, due to the origins of the concept of invariant control set (and to the fact that I followed the innumerous suggestions made by Professor Arnold), the questions treated here are essentially those motivated by the unique ergodicity problem of stochastic systems. This is reflected for instance by the emphasis put on counting the number of invariant control sets.

Despite that, I include some results on controllability and about the structure of invariant control sets which apparently are not related to stochastics. This is done as an attempt to regard the invariant control sets as genuine objects of control theory which should be considered independently of any exogenous application. In fact, if one studies control systems by analogy with the theory of topological dynamics, the (closed) invariant control sets appear as the analogues of the minimal invariant sets.

In this work attention is focussed on systems evolving on fibre bundles. Examples of such systems are provided by the coupling of an arbitrary system with a system evolving on a homogeneous space (see example E2.8 in §2). Another class of examples is obtained by taking the 'lifting' or 'prolongation' of a system on a manifold M to some fibre bundle associated to the bundle of k-th order frames of M (see examples E2.1, §2 and E3.1, §3 for first order cases). Coupled control systems appear in [2], [3] and [7], in connection with the study of linear systems driven by a 'real noise' and first order liftings as above are considered in relation to the Lyapunov exponents of a nonlinear stochastic differential equation in [5] and [9].

Let me outline the main technique used here to study invariant control sets on fibre hundles.

The fibre bundle $E \to M$ is viewed as a bunch of homogeneous spaces over the base manifold M, each fibre of the associated principal bundle $Q \to M$ - considered as Lie groups - acts on the corresponding fibre of $E \to M$. A control system $E\Sigma$ on E is 'lifted' to a control system $Q\Sigma$ on Q. The system $Q\Sigma$ defines subsemi-groups of the fibres of Q, and the invariant control sets of these semi-groups on the fibres of E are bunched together over the invariant control sets of the corresponding system Σ on M to form the invariant control sets of $E\Sigma$ on E.

In this approach, the invariant control sets of Σ on the base M are assumed as data and what remains to be done is to analyze invariant control sets of semi-groups on homogeneous spaces.

In this thesis, §§1,2 and 3 are intended to investigate the situations in which this technique works, while in §§4,5 and 6 invariant control sets of semi-groups on homogeneous spaces are considered. In a more detailed way, the contents are:

§0. Contains the basic terminology and assumptions. In this section the notions of controllability, unique controllability and index of controllability of a homogeneous space are introduced. I include also the statement of the orbit theorem of Stefan and Sussmann for future reference.

§1. Contains a theorem which is useful when dealing with the accessibility property of control systems. Two applications of this theorem are done. The first one is about controllability of measure preserving families of vector fields and is also included in §1. The other application is the one for which the theorem was designed: the systems on principal bundles of §2.

§2. Is to study control systems on principal bundles. The first step consists in the investigation of their orbits. These turn out to be sub-bundles. The second step is the analysis of the forward orbits, now under the assumption of transitivity. For future needs this second step is done for more general semi-groups than those generated by control systems. The properties required to these semi-groups are abstracted from the properties satisfied by the forward orbits of control systems.

Several examples are treated, e.g., the lifting of systems to the bundle of linear frames, trivial bundles (= coupled control systems) and equivariant fiberings of homogeneous spaces. This last one will be needed later in §6. Some of these are not specific examples so I call them special cases. Within these special cases consequences of the general results may appear. For instance, in example E2.10 I include proposition E2.10.1 which is a consequence of a previous result that works in the situation of example E2.10.

§3. The construction of invariant control sets on fibre bundles is done. Some examples are presented. These are mainly obtained from the examples of §2 and as there results are included within the examples. For instance proposition E3.3.1 which appears in E3.3 is needed throughout §§4,5 and 6.

§4. This section deals with invariant control sets on the boundaries of a semi-simple Lie group G. It is proved that a semi-group S with non void interior in G has a unique invariant control set on the boundaries of G and a characterization of this invariant control set is given in terms of the semi-simple elements in the interior of S. As a consequence of this characterization it is shown that the only semi-group of G which is controllable on its maximum boundary is G itself.

§5. Considers two semi-simple Lie groups $G \subset \bar{G}$ and a semi-group $S \subset G$ with non void interior in G. The question is about the closed invariant control sets of S on the boundaries of G. What is proved is that these invariant control sets are contained in the closed G-orbits and that inside these orbits there is a unique invariant control set of S. The motivation for considering this situation is twofold. Firstly, in applications Grassmannians and flag manifolds (= boundaries of $Sl(n,\mathbb{R})$) appear frequently. Also, the closed G-orbits are used in §6.

§6. Concerns the index of controllability of homogeneous spaces. An upper bound is given for the index of controllability of an arbitrary homogeneous space and in some nice situations they are effectively computed.

I finish this introduction by expressing once more my gratitude to Ludwig Arnold. This work is permeated by his influence and points of view.

§0 Basic Results, Concepts and Assumptions.

Control systems and their orbits:

A control system as is understood here is a family of vector fields Σ on a differentiable manifold M of class C^r, $1 \leq r \leq \omega$. Throughout M is assumed to be paracompact, Hausdorff, connected and of finite dimension n.

By a vector field on M we mean a C^{r-1}-local section of the tangent bundle TM of M, so that the domain of definition of $X \in \Sigma$ is usually an open subset of M.

When a control system Σ appears, it is tacitly assumed that

- Σ is everywhere defined on M in the sense that domains of definition of its elements cover M.
- The flow of local diffeomorphisms of $X \in \Sigma$ is of class C^r, the same class of differentiability as M.

The flow of local diffeomorphisms generated by the vector field X is denoted by X_t.

Associated to Σ, there is the group it generates or the system group G_Σ. This is the set of C^r-local diffeomorphisms of the type $X^1_{t_1} \circ \dots \circ X^k_{t_k}$ with $X^i \in \Sigma$ and the t_i's positive or negative reals.

Note that G_Σ is not a group in the strict sense because its elements are not always composable. G_Σ would be more properly a pseudogroup of local diffeomorphisms. However, to have G_Σ as a pseudogroup as defined for example in [32] or [36], the above set of local diffeomorphisms must be enlarged to a bigger one. Since this bigger set is not needed here, we do not bother to precise its construction and prefer to perpetrate the abuse of language of saying that G_Σ is a group of local diffeomorphisms. Sometimes this abuse is compensated by saying that G_Σ - or a 'group' of local diffeomorphisms like G_Σ - is a (pseudo) group.

These remarks are valid also to the one-parameter group (or flow) of local diffeomorphisms X_t generated by the vector field X .

If ϕ is a local diffeomorphism of M , its differential is denoted by ϕ_* or $d\phi$.

The <u>orbit</u> by Σ of $x \in M$ is the set $G_\Sigma(x) = \{\phi(x) : \phi \in G_\Sigma\}$. Σ is said to be <u>transitive</u> if for some (hence for all) $x \in M$, $G_\Sigma(x) = M$.

The relation 'y is in the orbit of x' is an equivalence relation in M . This equivalence relation partitions M into the orbits of Σ and is differentiable in the sense of the following theorem:

<u>Theorem 0.1</u> (Stefan [38], Sussmann [40]) : If Σ is a control system satisfying the above assumptions, then the partition of M into its orbits is a foliation with singularities.

This means that

a) Each orbit $G_\Sigma(x)$ is a quasi-regularly immersed submanifold of M (a leaf in the language of Stefan [38]. Our terminology comes from Varadarajan [43]). Thus $G_\Sigma(x)$ is a C^r-immersed submanifold of M with the property that if $\phi: N \to M$ is continuous from the locally connected topological space N and assume its values in $G_\Sigma(x)$, then $\phi : N \to G_\Sigma(x)$ is continuous w.r.t. the topology of $G_\Sigma(x)$.

b) Around every $x \in M$ there is a coordinate system $\psi : U \times V \subset \mathbb{R}^k \times \mathbb{R}^{n-k} \to M$ ($k = \dim G_\Sigma(x)$, $n = \dim M$, U and V open) such that $\psi(0,0) = x$ and in this coordinate system the orbits of Σ are the inverse images by the projection $U \times V \to V$ of their intersections with $\{0\} \times V$.

The tangent space $T_x G_\Sigma(x)$ is the subspace of $T_x M$ spanned by the set of vectors

$$\{\phi_* X(y) : \phi \in G_\Sigma , X \in \Sigma \text{ and } \phi(y) = x\} .$$

When $r = \omega$ and the vector fields in Σ is analytic (in this it is implicit that $X \in \Sigma$ is everywhere defined in M), $T_x\, G_\Sigma(x)$ is given by

$$LA(\Sigma)(x) = \{Z(x) \in T_x M : Z \in LA(\Sigma)\}$$

where $LA(\Sigma)$ stands for the Lie algebra of vector fields generated by Σ. //

From the characteristics of the orbits in this theorem, one sees immediately that if α_t is a differentiable curve in M starting at x then α_t stays in $G_\Sigma(x)$ iff its derivative α_t' is tangent to the orbit. In particular, a vector field Z in M preserves the orbits of Σ (i.e. $Z_t(y) \in G_\Sigma(y)$) iff Z is tangent to the orbits. These facts will be used later without any further comment.

The forward orbits:

The semi-group generated by Σ or the system semi-group S_Σ is defined by

$$S_\Sigma = \{X^1_{t_1} \circ \dots \circ X^k_{t_k} : X^i \in \Sigma\ ,\ t_i \geq 0\} \quad .$$

The previous remarks on the nature of G_Σ concerns also the nature of S_Σ . Sometimes we use the word (pseudo) semi-group to denominate an object like S_Σ .

The forward or positive orbit of $x \in M$ by Σ is the set $S_\Sigma(x) = \{\phi(x) : \phi \in S_\Sigma\}$.

Associated to the forward orbits there are the concepts of

Controllability: Σ is said to be controllable from $x \in M$ if $S_\Sigma(x) = M$, and Σ is controllable if it is controllable from every $x \in M$.

Approximate controllability: The same as controllability when

$S_\Sigma(x)$ is changed by $\mathrm{cl}\, S_\Sigma(x)$ where cl denotes closure.

Accessibility: Σ is accessible from $x \in M$ if $\mathrm{int}\, S_\Sigma(x) \neq \emptyset$, where int means interior. Σ is accessible or is said to have the accessibility property if it is accessible from every $x \in M$.

These notions were originated in relation to control systems (see [23], [24] and [39]). They depend only on the system semi-group S, so that we are allowed to use them for arbitrary (pseudo) semi-groups.

Concerning the accessibility property of Σ, let us quote the following result.

Theorem 0.2 ([23], [24]): In a smooth situation (i.e., $r = \infty$ and $X \in \Sigma$ is C^∞), Σ is accessible from $x \in M$ if (and only if in the analytic case) $\dim LA(\Sigma)(x) = \dim M$. //

Invariant Control Sets:

Given a (pseudo) semi-group S on M, an invariant control set of S or an S-i.c.s. is a set $C \neq \emptyset$ satisfying

i) For all $x \in C$, $\mathrm{cl}\, Sx = \mathrm{cl}\, C$.

ii) C is maximal with property i).

Note that by ii), two S-i.c.s's are either disjoint or identical. Also, if C is closed then it is an S-i.c.s. provided i) is satisfied.

The proofs of the following facts about invariant control sets are easily adapted from the corresponding proofs in [3] and [21].

1) If S has the accessibility property then every S-i.c.s. is closed. Let C be one of these sets. Then $\operatorname{int} C \neq \emptyset$ and $\operatorname{cl} \operatorname{int} C = C$.

2) If $C = \bigcap_{x \in M} \operatorname{cl} S(x)$ is not empty, it is the only possible closed S-i.c.s. . So that if S is accessible and $C \neq \emptyset$ then C is the unique S-i.c.s. .

3) If M is compact then for all $x \in M$, $\operatorname{cl} Sx$ contains a closed S-i.c.s. . Moreover, in case S is accessible, the number of S-i.c.s's in M is finite.

In contradiction to 3), if M is not compact, invariant control sets may not exist (the system $\Sigma = \{d/dx\}$ in $\mathbb{R}$ does not admit invariant control sets). Therefore the natural places where invariant control sets are to be considered are the compact manifolds.

A way of producing semi-groups S as above is to consider a Lie group G , to take S as a subsemi-group of G and view it as a semi-group of diffeomorphisms on a manifold in which G acts. Concerning these semi-groups, let us introduce the following.

<u>Definition</u>: Let G be a (not necessarily connected) Lie group, M a compact manifold in which G a cts transitively and put $M = G/L$, L a closed subgroup.

Let S be a subsemi-group of G with $\operatorname{int} S \neq \emptyset$ and which generates G in the sense that every element of G is a finite product of elements of S or $S^{-1} = \{g^{-1} : g \in S\}$.

Viewing S as a semi-group of diffeomorphisms of M , S is accessible, so that it has a finite number of invariant control sets. We call this number the index of controllability of S in $M = G/L$ and denote it by $ic(S,M)$.

The <u>index of controllability</u> of M as a homogeneous space of G is the supremum over the semi-groups S as above of $ic(S,M)$. It is denoted by $ic(M,G)$ or simply by $ic(G/L)$.

In case $ic(G/L) = 1$, G/L is said to be <u>unique controllable</u>.

G/L is said to be <u>controllable</u> in case every S as above is controllable in G/L .

<u>Remarks</u>: (1) Although $ic(S,M)$ is finite for every S with $\operatorname{int} S \neq \emptyset$ and S generating G , it is not automatic that $ic(M,G)$ is finite. This is however the case as is shown in §6.

(2) The class of semi-groups used to pattern the above definition is suggested by the semi-groups S_q which appears in §§2 and 3.

(3) If $L_1 \subset L$ is closed and normal in G , the action of G on G/L depends only on the action of G/L_1 on $(G/L_1)/(L/L_1)$, so that $ic(G/L) = ic((G/L_1)/(L/L_1))$.

§1. On the Accessibility Property of Control Systems.

Later on in sections 2 and 3 we shall be concerned with questions involving the forward orbit of a control system on a fibre bundle. These questions are mainly related to 'lifting' what is known about the projected system on the basis to the knowledge of some features of the system on the bundle. In this section we prove a theorem designed to deal with the accessibility property of such systems. As a consequence of this theorem it is also proved here a result about the controllability of measure preserving vector fields.

We start by introducing some terminology.

Suppose that $D = (X^1,\dots,X^k)$ is a finite sequence of vector fields on the manifold M and consider the map

$$\rho_D(\tau,x) = X^1_{t_1} \circ \dots \circ X^k_{t_k}(x) \tag{1.1}$$

with $\tau = (t_1,\dots,t_k) \in \mathbb{R}^k$. We don't assume that the vector fields X^j are complete or even that they are everywhere defined on M, so ρ_D is in general defined only on a subset of $M \times \mathbb{R}^k$.

Also, if $\rho_{D,x}$ stands for the mapping $\rho_{D,x}(\tau) = \rho_D(\tau,x)$ then $\rho_{D,x}$ is not necessarily defined for every $x \in M$ and its domain of definition may be a proper subset of $\mathbb{R}^k$. However, when talking about $\rho_{D,x}$ we assume that it is defined in an open subset $U \subset \mathbb{R}^k$ of the <u>control type</u> in the sense that if $(t_1,\dots,t_k) \in U$ and s is 0 or of the same sign as t_j then $(0,\dots,s,t_{j+1},\dots,t_k) \in U$ for every

$j = 1,\dots,k$. Thus U is connected and for every $\tau \in U$, $\rho_{D,x}(U)$ contains the trajectory induced by $\rho_{D,x}(\tau)$, i.e., the curve $\alpha:[0,T] \to M$, $T = |t_1| + \dots + |t_k|$, given by $\alpha(t) = X^j_s \circ X^{j+1}_{t_{j+1}} \circ \dots \circ X^k_{t_k}(x)$ if $|t_{j+1}| + \dots + |t_k| \le t \le |t_j| + \dots + |t_k|$, where $\pm s = t - (|t_{j+1}| + \dots + |t_k|)$ and the sign is taken according to the sign of t_j .

We denote $\tilde{D} = (X^k,\dots\dots,X^1)$ if D is as above and $\tilde{\tau} = (t_k,\dots,t_1)$ if $\tau = (t_1,\dots,t_k) \in \mathbb{R}^k$.

Given a system Σ and a point $x \in M$, define the rank of x w.r.t. Σ denoted by $\mathrm{rank}_\Sigma(x)$ to be the maximum of the ranks of $\rho_{D,x}$ at τ for all $D \subset \Sigma$ and $\tau = (t_1,\dots,t_k)$ positive, i.e., $t_1 \ge 0 ,\dots, t_k \ge 0$. Clearly, $\mathrm{rank}_\Sigma(x)$ is an integer valued lower semicontinuous function in M . Define also $\mathrm{rank}(\Sigma) = \max_{x \in M} \mathrm{rank}_\Sigma(x)$ and let us say that x is a <u>regular point</u> for Σ if $\mathrm{rank}_\Sigma(x) = \mathrm{rank}(\Sigma)$, whereas y is a <u>regular value</u> for Σ if $y = \rho_{D,x}(\tau)$ for some x and D with rank of $\rho_{D,x}$ at τ maximal. Of course, if $\mathrm{rank}_\Sigma(x) = \dim M$ then Σ is accessible from x . Sussmann [41] called Σ <u>normally accessible</u> from x if this condition is satisfied and showed that in case Σ has the accessibility property then it is normally accessible from every $x \in M$ (c.f. th. 4.1 in [41]).

Let R^+_Σ and R^-_Σ (or simply R^+ and R^-) be the set of regular points and regular values of Σ respectively. By the lower semicontinuity of rank_Σ , R^+ is open in M . R^- is also open as comes from the following lemma which establishes the relationship between R^+ and R^- and the corresponding sets for $-\Sigma = \{-X : X \in \Sigma\}$.

Lemma 1.1 : Suppose that $y = \rho_{D,x}(\tau)$. Then $dX^k_{-t_k} \circ \dots \circ dX^1_{-t_1}$ takes the image of $d(\rho_{D,x})_\tau$ into the image of $d(\rho_{-\tilde{D},y})_{\tilde{\tau}}$. In particular $\mathrm{rank}(\Sigma) = \mathrm{rank}(-\Sigma)$, $R^-_\Sigma = R^+_{-\Sigma}$, hence R^- is open, and y is a regular value for Σ iff it is a regular point for $-\Sigma$.

Proof: The image of $d(\rho_{D,x})_\tau$ is generated by the partial derivatives $(\partial\rho_{D,x}/\partial t_i)(\tau)$. But,

$$\frac{\partial\rho_{D,x}}{\partial t_i}(\tau) = dX^1_{t_1} \circ \dots \circ dX^{i-1}_{t_{i-1}}(X^i(z_i)) \tag{1.2}$$

where $z_i = X^i_{t_i} \circ \dots \circ X^k_{t_k}(x)$. So

$$dX^k_{-t_k} \circ \dots \circ dX^1_{-t_1} \left(\frac{\partial\rho_{D,x}}{\partial t_i}(\tau)\right) = -\frac{\partial\rho_{-\tilde{D},y}}{\partial t_i}(\tilde{\tau})$$

from which the lemma follows. //

It is readily seen from formula (1.2) that R^+ is invariant w.r.t. backward trajectories of Σ , i.e., if $x \in R^+$ then $S_{-\Sigma}(x) \subset R^+$, also R^- is invariant to forward trajectories of Σ , i.e., $S_\Sigma(x) \subset R^-$ if $x \in R^-$.

Denoting - for an open submanifold W of M- by $\Sigma|_W$ the set of restrictions $X|_W$ of vector fields $X \in \Sigma$, we state

Theorem 1.2 : Suppose that $R^+ \cap R^- \neq \emptyset$ and let $r = \mathrm{rank}(\Sigma)$.

Then there exists an integrable distribution $z \to \mathcal{D}_z \subset T_zM$, $z \in R^+ \cap R^-$, of constant dimension r in $R^+ \cap R^-$ such that if $X \in \Sigma$ then $X|_{R^+ \cap R^-}$ is tangent to $\mathcal{D}$.

The maximal integral submanifolds of $\mathcal{D}$ are then invariant to forward and backward trajectories of $\Sigma|_{R^+ \cap R^-}$ and every orbit of this system is contained in some maximal integral submanifold of $\mathcal{D}$.

In particular, if $R^+ \cap R^- = M$ and Σ is transitive then $\mathrm{rank}(\Sigma) =$ $= \dim M$ and Σ has the accessibility property.

Proof: Let $z \in R^+ \cap R^-$. Since $z \in R^-$ there exist $x \in R^+$, $E = (X^1, \dots, X^k) \subset \Sigma$ and positive τ_0 such that $z = \rho_{E,x}(\tau_0)$ and $\rho_{E,x}$ has rank r at τ_0. Maximality of r entails that for some neighbourhood U_0 of τ_0, $\rho_{E,x}(U_0)$ is a r-dimensional submanifold of M.

Denote this submanifold by N_z and put $\mathcal{D}_z = T_zN_z$.

We claim that $\mathcal{D}_z$ is independent of the choice of E, x and τ_0.

Indeed, since z is also in R^+ there exists $F = (Y^1, \dots, Y^m) \subset \Sigma$ such that $\rho_{F,z}$ has rank r at some σ_0 in its domain of definition. Put $y = \rho_{F,z}(\sigma_0)$.

Let $D = (F,E)$ and make the concatenation

$$\rho_{D,x}(\tau,\sigma) = Y^1_{s_1} \circ \dots \circ Y^m_{s_m} \circ X^1_{t_1} \circ \dots \circ X^k_{t_k}(x) .$$

Applying formula (1.2) to this D one sees that $\rho_{D,x}$ has rank r at (τ_0,σ_0) so there are open sets $U \subset \mathbb{R}^k$, $V \subset \mathbb{R}^m$ with $(\tau_0,\sigma_0) \in U \times V$ and such that $\rho_{D,x}$ has rank r in $U \times V$.

Therefore $\rho^{-1}_{D,x}(y)$ is a submanifold of $U \times V$ whose tangent space at (τ,σ) is Ker $d(\rho_{D,x})_{(\tau,\sigma)}$. But, from (1.2)

$$\frac{\partial\rho_{D,x}}{\partial s_i}(\tau_0,\sigma_0) = \frac{\partial\rho_{F,z}}{\partial s_i}(\sigma_0) ; \quad i = 1,\dots, m ,$$

and since r is the maximum of the ranks, these partial derivatives generate the r-dimensional subspace

$$\text{span}(\frac{\partial\rho_{D,x}}{\partial t_i}(\tau_0,\sigma_0) ; \frac{\partial\rho_{D,x}}{\partial s_j}(\tau_0,\sigma_0) ; \; i = 1,\dots,k ; j = 1,\dots,m)$$

so that for every $v \in \mathbb{R}^k$ there exists $w \in \mathbb{R}^m$ with $d(\rho_{D,x})_{(\tau_0,\sigma_0)}(v,w) = 0$. It follows that the tangent space of $\rho^{-1}_{D,x}(y)$ is projected onto $\mathbb{R}^k$ by the projection $\pi:(\tau,\sigma) \to \tau$, hence π restricted to $\rho^{-1}_{D,x}(y)$ is locally a submersion. The implicit function theorem implies then the existence of a section $\mu:U' \subset U \to V$, $\pi\circ\mu = id_{U'}$, with $\mu(\tau_0) = \sigma_0$ s.t. $\rho_{D,x}(\tau,\mu(\tau)) = y$, $\tau \in U'$.

If $\mu = (\mu_1,\dots,\mu_m)$ let $-\tilde{\mu} = (-\mu_m,\dots,-\mu_1)$. Then

$$\rho_{-\tilde{F},y} \circ (-\tilde{\mu})(\tau) = Y^m_{-\mu_m(\tau)} \circ\dots\circ Y^1_{-\mu_1(\tau)}(y) =$$

$$= Y^m_{-\mu_m(\tau)} \circ\dots\circ Y^1_{-\mu_1(\tau)}(\rho_{D,x}(\tau,\mu(\tau))) =$$

$$= X^1_{t_1} \circ\dots\circ X^k_{t_k}(x)$$

$$= \rho_{E,x}(\tau)$$

for every $\tau = (t_1,\dots,t_k) \in U'$.

Therefore, by shrinking U' and V if necessary we can find open sets V' and U'' with $-\sigma_0 \in V'$, $\tau_0 \in U'' \subset U'$ such that $\rho_{E,x}(U'') \subset \rho_{-\tilde{F},y}(V')$ and $\rho_{-\tilde{F},y}$ has rank r at every point of V' .

It follows that $\mathcal{D}_z = d(\rho_{-\tilde{F},y})\sigma_0$ regardless the choice of E , x and τ_0 as claimed.

The assignment $z \to \mathcal{D}_z$ defines then a n-dimensional distribution in $R^+ \cap R^-$, which by construction is integrable.

To conclude the proof of the theorem we need only to verify that $X \in \Sigma$ is tangent to $\mathcal{D}$. Suppose not and let $N_z = \rho_{D,x}(U)$ be as above with X not tangent to N_z at z . Then the map $X_s \circ X^1_{t_1} \circ\dots\circ X^k_{t_k}(x)$, $|s| \le \varepsilon$, has rank $r+1$ at $(0,\tau_0)$ which contradicts the assumption. //

Remark: The orbits of $\Sigma\big|_{R^+ \cap R^-}$ may be properly contained in the maximal integral manifolds of $\mathcal{D}$ as is shown by the following example.

Example 1.1 : Take M to be $\mathbb{R}^3$ and $\Sigma = (X_1, X_2, X_3)$ with $X_1 = \frac{\partial}{\partial x}$; $X_2 = \frac{\partial}{\partial y}$ restricted to the open set $\{(x,y,z) \in \mathbb{R}^3 : -1 < x < 1\}$; $X_3 = \frac{\partial}{\partial z}$ restricted to $\{(x,y,z) : x < -1 \text{ or } x > 1\}$.

Then $\operatorname{rank}(\Sigma) = 3$, $R^+ \cap R^- = \{(x,y,z) : -1 < x < 1\}$ and the orbits of $\Sigma\big|_{R^+ \cap R^-}$ have dimension 2 so they are properly contained in $R^+ \cap R^-$ which is the only integral manifold of $\mathcal{D}$. //

As a consequence of the above theorem we have the following extension of a theorem by C. Lobry [25].

Theorem 1.3 : Suppose that the vector fields in Σ are defined everywhere in M , are complete and preserve a finite Borel measure m positive on non void open sets.

Assume Σ transitive. Then

a) $R^+ \cap R^-$ is dense in M .

b) $\operatorname{rank}(\Sigma) = \dim M$.

c) If $x \in R^+ \cap R^-$ then $\operatorname{cl} S_\Sigma(x) = M$.

d) $R^+ \cap R^-$ is connected.

We first prove

Lemma 1.4 : With the assumptions as in the theorem let U be an open set in M invariant to forward trajectories of Σ . Then U is dense in M .

Proof: In view of transitivity it is sufficient to show that cl U is invariant to forward and backward trajectories of Σ . Invariance to forward trajectories follows immediately from the corresponding invariance of U .

As to the backward case, let $x \in \text{cl } U$ and assume that for some $X \in \Sigma$ there exists an open V with $V \cap \text{cl } U = \emptyset$ and $X_{-t}(x) \in V$, $t > 0$. Let F be the recurrent set of X_t in $X_t(V)$ (c.f. Halmos [13]). Then if $y \in F$, $X_T(y) \in X_t(V)$ for some $T > t$.

Since m is positive on open sets and $m(F) = m(X_t(V))$, F is dense in $X_t(V)$, so that it is possible to take $y \in F \cap U$. We have then that $X_{T-t}(y) \in V \cap U$ contrary to the choice of V . //

Corollary 1.5 (Lobry [25]) : Σ is controllable from every $x \in M$ provided Σ and $-\Sigma$ have the accessibility property; in particular provided Σ satisfies the Lie algebra rank condition.

Proof: Given $x,y \in M$, we have by the lemma above that $S_\Sigma(x)$ is dense and since $S_{-\Sigma}(y)$ has non void interior, $S_\Sigma(x) \cap S_{-\Sigma}(y) \neq \emptyset$ and x is controllable into y . //

<u>Note</u>: Controllability of Σ is not achieved under accessibility of Σ alone as is shown by example 1.2 below.

<u>Proof of Th. 1.3</u> : From lemma 1.4 and invariance of R^- w.r.t. forward trajectories of Σ we have that R^- is dense in M. Applying the same reasoning to $-\Sigma$ we see that R^+ is also dense and thus a).

Lemma 1.4 also shows that b) implies c).

To prove b) we make use of lemma 5.2 in [40] and transitivity of Σ to get, for any $x,z \in R^+ \cap R^-$ a sequence $D = (X^1,\ldots,X^k) \subset \Sigma$ such that $z = \rho_{D,x}(\bar{\tau})$ and $\rho_{D,x}$ has rank n (the dimension of M) at $\bar{\tau} = (\bar{t}_1,\ldots,\bar{t}_k) \in \mathbb{R}^k$. The image of $\rho_{D,x}$ is not necessarily contained in $R^+ \cap R^-$.

Let us show that it is possible to change x without changing the rank at $\bar{\tau}$ and in such a way that $\rho_{D,x}(W) \subset R^+ \cap R^-$ for some W of the control type in $\mathbb{R}^k$ with $\bar{\tau} \in W$.

Let $V_0 \subset R^+ \cap R^-$ be open with $x \in V_0$ and such that if $y \in V_0$ then $\rho_{D,y}$ has rank n at $\bar{\tau}$. If we let F_0 denote the recurrent set of $X^k_{\bar{t}_k}$ on V_0 then for $y_0 \in F_0$ it is possible to find positive integers p_1 and p_2 such that $X^k_{p_1\bar{t}_k}(y_0) \in V_0$ and $X^k_{-p_2\bar{t}_k}(y_0) \in V_0$.

The invariance of R^+ to backward trajectories, of R^- to forward trajectories and the fact that $V_0 \subset R^+ \cap R^-$ imply then that $X^k_t(y_0)$

for t between 0 and $\bar{t}_k$ is contained in $R^+ \cap R^-$.

Now, put $F_1' = X^k_{\bar{t}_k}(F_0)$. Then $F_1' \subset R^+ \cap R^-$. Let $F_1 \subset F_1'$ with $m(F_1) = m(F_1') = m(F_0) = m(V_0) > 0$ be the recurrent set of $X^{k-1}_{\bar{t}_{k-1}}$ in F_1' . If $y_1 \in F_1$ then the same reasoning as above shows that $X^{k-1}_t(y_1)$ stays in $R^+ \cap R^-$ for t between 0 and $\bar{t}_{k-1}$, so that the trajectory of Σ induced by $\rho_{D^{k-1},X^k_{-\bar{t}_k}(y_1)}(\bar{t}_{k-1},\bar{t}_k)$, $D^{k-1} = (X^{k-1},X^k)$, is contained in $R^+ \cap R^-$.

Proceeding this way we will finally find $\bar{x} \in V_0$ such that the trajectory of Σ induced by $\rho_{D,\bar{x}}(\bar{\tau})$ is entirely contained in $R^+ \cap R^-$. It is then immediate the construction of W of the control type with $\bar{\tau} \in W$ and $\rho_{D,\bar{x}}(W) \subset R^+ \cap R^-$.

Since $\bar{x} \in V_0$, we have from the choice of V_0 that the rank of $\rho_{D,\bar{x}}$ at $\bar{\tau}$ is n as desired.

The sequence D can then be taken in $\Sigma\big|_{R^+\cap R^-}$. Let $I(\bar{x})$ be the maximal integral manifold of the distribution $\mathcal{D}$ of Th. 1.2. Then $I(\bar{x})$ is invariant by $\Sigma\big|_{R^+\cap R^-}$ hence $\rho_{D,x}(\bar{\tau}) \subset I(\bar{x})$ and $\dim I(\bar{x}) = n$. Therefore $\operatorname{rank}(\Sigma) = \dim I(\bar{x}) = n$ and we have b).

Finally, to see d) take V_0 small enough in order that $\rho_{D,\bar{\tau}}(V_0)$ is contained in some connected neighbourhood of z in $R^+ \cap R^-$. (Here $\rho_{D,\bar{\tau}}(x) = \rho_D(\bar{\tau},x))$. //

<u>Example 1.2</u> : This example is to show that in the situation of the theorem above it is not true in general that $R^+ \cap R^- = M$ neither that Σ is controllable.

Take M to be the two torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$.

The measure will be the canonical one induced by the volume element $dx \wedge dy$ in $\mathbb{R}^2$. The hamiltonian vector fields $X_H = \frac{\partial H}{\partial y}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial y}$, H a function on T^2 , are measure preserving.

Consider the functions

$H_1(x,y) = \cos 2\pi x \sin 2\pi y$; $H_2(x,y) = \sin 2\pi x \sin 2\pi y$

$H_3(x,y) = \cos 2\pi y$

$H_4(x,y) = H_4(x)$ a non-zero C^∞-function of x alone with support contained in the interval $\frac{5}{8} \le x \le \frac{7}{8}$.

$H_5(x,y) = H_5(y)$ non-zero with support contained in the interval $\frac{1}{4} \le y \le \frac{3}{4}$ and with $\frac{dH_5}{dy}(\frac{1}{2}) \ne 0$.

Let $X_i = X_{H_i}$, $i = 1,\dots,5$ and take

$$\Sigma = \{X_1 , \pm X_2 , \pm X_3 , \pm X_4 , \pm X_5\} .$$

Then $\pm X_4$ makes the link between the slices $S^1 \times \{y\}$, $y \in S^1$, while $\pm X_3$ is transitive on the slices $S^1 \times \{y\}$ for $y \ne 0$ and $y \ne \frac{1}{2}$.

$\pm X_5$ is transitive on $S^1 \times \{\frac{1}{2}\}$ and $\{X_1, \pm X_2\}$ is transitive on $S^1 \times \{0\}$, so Σ is transitive.

Let $C = \{(x,y) : 0 \le x \le \frac{1}{2}$ and $y = 0\}$. Then the only way of leaving C is by using $-X_1$, so C is invariant to forward trajectories of Σ and we have

$$R^+ = T^2 - C \; ; \; R^- = T^2 \text{ and } R^+ \cap R^- = T^2 - C \; .$$

$$\Sigma \text{ is controllable from } (x,y) \in T^2 - C$$

$$-\Sigma \text{ is controllable from } (x,y) \in C \; ,$$

however $-\Sigma$ is only approximate controllable from $(x,y) \notin C$, i.e., $\mathrm{cl}\, S_{-\Sigma}(x,y) = M$ and $S_{-\Sigma}(x,y) \neq M$. Σ is not accessible from $(x,y) \in C$ but $-\Sigma$ is accessible from every point of T^2.

Also, $T^2 - C$ is an invariant control set of Σ which is not invariant w.r.t. forward trajectories of Σ. //

§2. Control Systems on Principal Bundles.

This section is devoted to the study of some properties of the orbits and forward orbits of control systems on principal bundles.

Let $Q(M,G)$ be such a bundle with Q the total space, M the base and G the structure group. The canonical projection of Q onto M will be denoted by $\pi_Q : Q \to M$, whereas $R_a(q) = qa$, $a \in G$, $q \in Q$, stands for the right action of G on Q.

We denote a control system on Q by $Q\Sigma$ and the vector fields in it by QX.

The control systems on Q to be considered here are those families of vector fields consisting of infinitesimal automorphisms of the bundle Q. These are vector fields invariant by the right action of G, $R_{a*}(QX) = QX$, and defined on open sets of the type $\pi_Q^{-1}(U)$, U open in M. For such vector fields $X = \pi_{Q*}(QX)$ makes sense so that a family $Q\Sigma$ of infinitesimal automorphisms gives rise to a system $\Sigma = \pi_Q(Q\Sigma) = \{\pi_{Q*}(QX) = X : QX \in Q\Sigma\}$.

In the sequel unless otherwise specified the systems will be as above, so that $QX \in Q\Sigma$ satisfies

$$R_{a*}(QX) = QX \, , \quad a \in G \, . \tag{2.1}$$

The orbits of $Q\Sigma$

If $X = \pi_{Q*}(QX)$ then $\pi_Q \circ (QX)_t = X_t \circ \pi_Q$ which implies that $\pi_Q \, G_{Q\Sigma}(q) = G_\Sigma(\pi_Q(q))$, $q \in Q$, where Σ is the system on M defined above. So that the orbits of $Q\Sigma$ project down onto the orbits of Σ which are submanifolds of M. We show here that $G_{Q\Sigma}(q)$ is a principal bundle over $G_\Sigma(\pi_Q(q))$.

Let us begin by showing the existence of local sections.

Lemma 2.1 : For $q \in Q$ let $x = \pi_Q(q)$. Then there exists a (differentiable) local section $\mu : V \to G_{Q\Sigma}(q)$, V open in $G_\Sigma(x)$ with $x \in V$, $\pi_Q \circ \mu = \mathrm{id}_V$, such that $\mu(x) = q$.

Proof: As was already remarked, there are $D = (X^1, \ldots, X^k) \subset \Sigma$, $x_1 \in G_\Sigma(x)$ and $\tau^0 \in \mathbb{R}^k$ such that $x = \rho_{D,x_1}(\tau^0)$ and ρ_{D,x_1} has rank $m = \dim G_\Sigma(x)$ at τ^0.

Around τ^0, ρ_{D,x_1} is then a submersion so by the implicit function theorem there exists a map ϕ_1 defined in a neighbourhood U of $0 \in \mathbb{R}^m$ such that $\phi = \rho_{D,x_1} \circ \phi_1 : U \to V$ is a diffeomorphism between U and the open set $V \subset G_\Sigma(x)$ with $x \in V$, and $\phi(0) = x$.

Let $QD = (QX^1, \ldots, QX^k) \subset Q\Sigma$ be such that $\pi_{Q*}(QX^i) = X^i$, $i = 1, \ldots, k$; and put $q_1 = \rho_{\widetilde{QD},q}(-\tilde{\tau}^0)$. Then $x_1 = \pi_Q(q_1)$ and $\rho_{D,x_1} = \pi_Q \circ \rho_{QD,q_1}$.

The desired section is then given by

$$\mu = \rho_{QD,q_1} \circ \phi_1 \circ \phi^{-1} : V \to G_{Q\Sigma}(q) .$$

In fact, μ is a section because

$\pi_Q \circ \mu = \pi_Q \circ \rho_{QD,q_1} \circ \phi_1 \circ \phi^{-1} = \rho_{D,x_1} \circ \phi_1 \circ \phi^{-1} = id_V$.

Moreover, $\mu(x) = \rho_{QD,q_1} \circ \phi_1 \circ \phi^{-1}(x) = \rho_{QD,q_1}(\tau^0) = q$. //

Let us look now at the structure group.

From (2.1) it is readily seen that R_a , $a \in G$ intertwines the trajectories of $Q\Sigma$, so that for any $q \in Q$, $R_a(G_{Q\Sigma}(q)) = G_{Q\Sigma}(qa)$, i.e., R_a intertwines also the orbits of $Q\Sigma$ viewed as subsets of Q . Observe that since $G_{Q\Sigma}(q)$ is quasi-regularly immersed in Q , R_a actually establishes a diffeomorphism between $G_{Q\Sigma}(q)$ and $G_{Q\Sigma}(qa)$ so that all the orbits crossing a fixed fibre Q_x of Q are diffeomorphic.

This means that the orbits over $G_\Sigma(x)$ form a foliation (without singularities) of the principal bundle $Q|_{G_\Sigma(x)} = \pi_Q^{-1}(G_\Sigma(x))$. We remark the following quick way of seeing that $Q|_{G_\Sigma(x)}$ is in fact a principal bundle over $G_\Sigma(x)$: view $\pi_Q^{-1}(G_\Sigma(x))$ as an orbit of the family of vector fields in Q obtained by increasing $Q\Sigma$ by all the vertical vector fields satisfying (2.1). Local triviality follows from lemma 2.1 and the other properties are easily checked.

Now, for $q \in Q$ set

$$G_q = \{a \in G : R_a(G_{Q\Sigma}(q)) = G_{Q\Sigma}(qa) = G_{Q\Sigma}(q)\} .$$

Then G_q is a subgroup of G . In fact, if $a,b \in G_q$ then $R_{ab}(G_{Q\Sigma}(q)) = R_b \circ R_a(G_{Q\Sigma}(q)) = G_{Q\Sigma}(q)$, and $R_{a^{-1}}(G_{Q\Sigma}(q)) =$ $= R_{a^{-1}}(G_{Q\Sigma}(qa)) = G_{Q\Sigma}(q)$. Also, $a \in G_q$ iff $qa \in G_{Q\Sigma}(q)$ so that if we identify the fibre Q_x , $x = \pi_Q(q)$, through q with G by $qa \to a$, G_q is given by $G_{Q\Sigma} \cap Q_x$.

From the following lemma it will be proved that G_q is a Lie subgroup of G .

<u>Lemma 2.2</u> : Let $\phi : N \to M$ be an injective immersion with M and N paracompact and suppose that $\mathcal{D} : y \in M \to \mathcal{D}_y \subset T_yM$ is a distribution without singularities in M such that the dimension of $\phi_*(T_xN) \cap \mathcal{D}_{\phi(x)}$ is constant as a function of $x \in N$.

Let I be a maximal integral submanifold of $\mathcal{D}$.

Then $\phi^{-1}(I)$ is a quasi-regular immersed submanifold of N .

<u>Proof</u>: Take $x \in \phi^{-1}(I)$. Locally the situation assumes the following aspect: N is a neighbourhood U of 0 in some euclidean space, $x = 0$, $M = U_1 \times U_2$; U_1 , U_2 open connected subsets in convenient euclidean spaces, $\phi(x) = (0,0) \in U_1 \times U_2$ and the integral manifolds are the slices $\pi_2^{-1}(z)$, with $z \in U_2$ and π_2 the projection onto the second coordinate.

Also, the assumption on ϕ implies that $\pi_2 \circ \phi : U \to U_2$ is of constant rank so by shrinking U and U_2 if necessary, we can assume further that $\pi_2 \circ \phi$ is the restriction to U of a linear map. Therefore, inside U $\phi^{-1}(I) = (\pi_2 \circ \phi)^{-1}(I \cap (\{0\} \times U_2))$ is a union of parallel affine subspaces in U .

However, I is an immersed connected submanifold of M , which is paracompact, hence I is separable and since inside $U_1 \times U_2$ it is the union of the slices $\pi_2^{-1}(z)$, $z \in I \cap (\{0\} \times U_2)$, and these slices are open in I , we conclude that $I \cap (\{0\} \times U_2)$ is at most countable.

It follows that inside U , $\phi^{-1}(I)$ is the union of at most denumerable affine parallel subspaces. From this one can see easily that $\phi^{-1}(I)$ is in fact quasi-regular. //

Corollary 2.3 : G_q is a Lie subgroup of G .

Proof: Apply the previous lemma to $N = Q_x$, $\mathcal{D}$ the distribution in $Q|_{G\Sigma(x)}$ whose integral manifolds are the orbits of $Q\Sigma$ and ϕ the injection $Q_x \to Q|_{G\Sigma(x)}$. Then G_q is a quasi-regular submanifold of G and hence a Lie subgroup. //

Summarizing we have the following characterization of the orbits of $Q\Sigma$.

Theorem 2.4 : For each $q \in Q$, $G_{Q\Sigma}(q)$ is a G_q - principal subbundle of $Q|_{G_{Q\Sigma}(x)}$, $x = \pi_Q(q)$. If $q' = qa$ then $G_{q'} = a^{-1}G_qa$ and $G_{Q\Sigma}(q') = R_a(G_{Q\Sigma}(q))$.

Proof: The fact that $G_{Q\Sigma}(q)$ is a subbundle follows from Lemma 2.1 and corollary 2.3.

To see the conjugacy of the groups, take $b \in G_{q'}$. Then $qab = q'b \in G_{Q\Sigma}(q') = R_a(G_{Q\Sigma}(q))$. Hence $qab = qca$ for some $c \in G_q$ so that $b = a^{-1}ca$. //

This characterization of the orbits is valid for families of vector fields that satisfy (2.1) and thus are kept fixed by R_{a*}, $a \in G$. However, there are interesting examples of systems on principal bundles whose vector fields are not invariant by the action of G but in which the system itself is invariant in the sense that for all $a \in G$

$$R_{a*}(QX) \in Q\Sigma \qquad (2.2)$$

if $QX \in Q\Sigma$ (see examples E2.5 and E2.6 below). Let us verify that theorem 2.4 is also valid for these systems.

<u>Proposition 2.5</u> : Suppose that $Q\Sigma$ satisfies (2.2). Then there exists $Q\Sigma_1$ satisfying (2.1) having the same orbits as $Q\Sigma$.

<u>Proof</u>: Consider the family $Q\Sigma_2$ of all those local vector fields QX on Q such that if $QX(q)$ is defined then it is tangent to the orbit $G_{Q\Sigma}(q)$. The solution of a vector field in $Q\Sigma_2$ starting in an orbit of $Q\Sigma$ never leaves it, hence the orbits of $Q\Sigma$ are invariant by $G_{Q\Sigma_2}$. However, $Q\Sigma \subset Q\Sigma_2$ so that $Q\Sigma$ and $Q\Sigma_2$ have the same orbits. The advantage of $Q\Sigma_2$ w.r.t. $Q\Sigma$ is that the set $Q\Sigma_2(q) = \{QX(q) : QX \in Q\Sigma_2\}$ spans the tangent space to $G_{Q\Sigma}(q)$ as can be easily seen from the construction of the tangent space to the orbits (c.f. Stefan [38] or Sussmann [40]).

Now, take a coordinate system of Q in which it is written as a product $U \times G$, and for $QX \in Q\Sigma_2$ define $\overline{QX}(x,a) = R_{a*}QX(x,1)$, $x \in U$,

$a \in G$. Then $\overline{QX}$ satisfies (2.1) and since $QX(x,1)$ is tangent to $G_{Q\Sigma}(x,1)$ and R_a is a diffeomorphism between the orbits, $\overline{QX}$ is tangent to the orbits of $Q\Sigma$. Therefore, $(\overline{QX})_t(q) \in G_{Q\Sigma}(q)$ for all q for which the expression is defined.

Varying this construction through all coordinate system as above and all $QX \in Q\Sigma_2$ we get a family $Q\Sigma_1$ of vector fields satisfying (2.1) which have the same orbits as $Q\Sigma$. //

<u>The Forward Orbits of $Q\Sigma$</u>

We restrict $Q\Sigma$ to its orbits which in view of theorem 2.4 amounts to assuming that $Q\Sigma$ is transitive on Q .

Denoting as before by Σ the system on M obtained by projecting $Q\Sigma$, we have

<u>Proposition 2.6</u> : If $Q\Sigma$ is transitive on Q then it satisfies the accessibility property provided Σ is controllable.

<u>Proof</u>: $R^-_{Q\Sigma}$ is invariant w.r.t. forward trajectories of $Q\Sigma$ so controllability of Σ implies that $R^-_{Q\Sigma}$ meets every fibre of Q . The invariance (2.1) implies then that $R^-_{Q\Sigma} = Q$.

If Σ is controllable then $-\Sigma$ is also controllable so using the same reasoning we conclude that $R^+_{Q\Sigma} = M$. Accessibility follows then from theorem 1.2. //

<u>Remark</u>: The affirmation in this proposition is also valid for systems satisfying the relaxed invariance condition (2.2). For this situation

we don't have Σ but the same proof works if instead of controllability of Σ we assume that $Q\Sigma$ is "fibre controllable" in the sense that every fibre of Q can be reached from all $q \in Q$. Actually, we can weaken even more the condition on $Q\Sigma$ by requiring only that its forward orbits are invariant by R_a, i.e., $R_a(S_{Q\Sigma}(q)) \subset S_{Q\Sigma}(qa)$ all $a \in G$, $q \in Q$. To see this use the following trick: enlarge $Q\Sigma$ to a system $Q\Sigma_e$ formed by all those local vector fields QX in Q such that $(QX)_t(q) \in S_{Q\Sigma}(q)$ for small $t \geq 0$ and q in the domain of QX. Then $Q\Sigma_e$ satisfies (2.2) and has the same forward orbits as $Q\Sigma$. //

In the sequel we will need only the semi-group property of $S_{Q\Sigma}$ so we will work more generally with a semi-group S_Q of local diffeomorphisms of Q commuting with R_a, $a \in G$ and defined in open subsets of the type $\pi_Q^{-1}(U)$, U open in M. S_Q induces a semi-group S_M of local diffeomorphisms of M. Also, for this general situation we do not need to assume M connected. This is because we usually deal with the following hypothesis on S_Q which can be satisfied in arbitrary M.

a) S_Q is accessible, i.e., for all $q \in Q$ int $S_Q(q) \neq \emptyset$.

HS b) S_M is controllable, i.e., $S_M(x) = M$ for all $x \in M$.

c) The (pseudo) group G_Q generated by S_Q is transitive on Q.

Note: In case Q is connected over the connected components of M a) and b) imply c). In fact, from a) $q \in \text{int}\, G_Q(q)$ for every $q \in Q$, hence G_Q is transitive over the connected components of M and b) implies its transitivity. //

To analyze the orbits of S_Q let us define for $q \in Q$ the set

$$S_q = S_Q(q) \cap Q_x , \quad x = \pi_Q(q) .$$

Through the identification of Q_x with G via $a \in G \to qa \in Q_x$, we can view S_q as a subset of G and as such we have

$$S_q = \{a \in G \; ; \; R_a(S_Q(q)) \subset S_Q(q)\} . \qquad (2.3)$$

Indeed, if $R_a(S_Q(q)) \subset S_Q(q)$ then $qa \in S_Q(q)$ so $a \in S_q$. Conversely, given $a \in S_q$ and $q' = Q\phi(q) \in S_Q(q)$, $Q\phi \in S_Q$, we have $q'a = Q\phi(qa) \in S_Q(q)$, hence $R_a(S_Q(q)) \subset S_Q(q)$.

From (2.3) it follows immediately that S_q is a subsemi-group of G .

Changing q by $q' = qa$ we get $S_{q'} = a^{-1}S_q a$. To see this take $c \in S_{q'}$ and put $q'c = Q\phi(q')$, $Q\phi \in S_Q$. Then $qac = Q\phi(qa) = Q\phi(q)a$ so that $qaca^{-1} = Q\phi(q)$ and $c \in a^{-1}S_q a$. The reverse can be seen by writing $q = q'a^{-1}$.

Proposition 2.7 : If S_Q satisfies HS then S_q has non void interior in G .

Proof: Choose $q' \in \text{int}\, S_Q(q)$. By controllability of S_M there exists $Q\phi \in S_Q$ with $Q\phi(q') \in Q_x$, $x = \pi_Q(q)$. Thus

$$\text{int}\, S_q \supset \text{int}\, Q\phi(S_Q(q)) \cap Q_x \neq \emptyset . \quad //$$

In case G is connected this proposition implies that S_q generates G in the sense that all $a \in G$ is a finite product of elements in $S_q \cup S_q^{-1}$. Let us see that the same thing happens for arbitrary G .

<u>Proposition 2.8</u> : Under HS S_q generates G .

<u>Proof</u>: Let $gen(S_q)$ stand for the group generated by S_q .

We shall prove first

(=) "If $Q\phi \in S_Q$ and $q_2 = Q\phi(q_1)$ then $gen(S_{q_1}) = gen(S_{q_2})$."

Take $a \in S_{q_2}$ and $Q\psi \in S_Q$ that maps the fibre through q into the fibre through q_1 .

Then $q_2a \in S_Q(q_2) \subset S_Q(q_1)$ and if we define b by $Q\psi(q_2a) = q_1b$ then $q_1b \in S_Q(q_1)$ so $b \in S_{q_1}$.

We have,

$$(Q\phi)^{-1}\circ(Q\psi)^{-1}(q_1b) = (Q\phi)^{-1}(q_2a) = q_1a$$

and we have also

$$(Q\phi)^{-1}\circ(Q\psi)^{-1}(q_1b) = (Q\phi)^{-1}\circ(Q\psi)^{-1}(q_1)b$$
$$= q_1\bar{a}b$$

for some $\bar{a} \in S_{q_1}^{-1}$. Indeed using the previous identifications it is easily seen that $S_q^{-1} = S_Q^{-1}(q) \cap Q_x$ where $S_Q^{-1} = \{(Q\phi)^{-1} : Q\phi \in S_Q\}$.

It follows that $a = \bar{a}b \in gen(S_{q_1})$ and $gen(S_{q_2}) \subset gen(S_{q_1})$.

Applying the same reasoning to $(Q\phi)^{-1} \in S_Q^{-1}$, we see that $gen(S_{q_1}^{-1}) \subset gen(S_{q_2}^{-1})$ so that $gen(S_{q_1}) = gen(S_{q_2})$ which proves (*).

Now, given $Q\psi \in S_Q$ with q in the domain of $(Q\psi)^{-1}$, choose $Q\phi \in S_Q$ that maps q into the fibre of $(Q\psi)^{-1}(q)$. Then $Q\psi \circ Q\phi(q) = qa$ for some $a \in S_q$ so that $(Q\psi)^{-1}(q) = Q\phi(q)a^{-1}$, i.e., $(Q\psi)^{-1}(q)$ can be "corrected" into $S_Q(q)$ by a^{-1}, which belongs to $gen(S_q)$.

Using this correction and (*) successively, and taking into account that G_Q is transitive on Q, we find that every element of G can be written as bc with $b \in S_q$ and $c \in gen(S_q)$, i.e., $G \subset gen(S_q)$ which proves the proposition. //

Remark: The difference in the construction of G_q made previously and of $gen(S_q)$ can be expressed by saying that while G_q is constructed by means of trajectories having positive and negative times indistinctly, to have $gen(S_q)$ in turn we must first return to the fibre of q using only positive times - for example - in order to be able to use negative times. //

Finally, we have the following controllability cases.

Proposition 2.9 : Assume HS. Then

i) If $S_q = G$ for some q then S_Q is controllable.

ii) If G is compact then S_Q is controllable.

Proof: i) depends only on b) of HS: take $q' \in Q$, then by controllability of S_M there exists $Q\phi \in S_Q$ with $Q\phi(q) = q'a$ for some $a \in G$. By the assumption there exists $Q\psi \in S_Q$ with $Q\psi(q) = qa^{-1}$. So that $q' = Q\phi(qa^{-1}) = Q\phi \circ Q\psi(q)$ and S_Q is controllable.

ii) By proposition 2.7 int $S_q \neq \emptyset$ so S_q contains the identity component G_0 of G (c.f. [20] or [34]) and since S_q generates G, $S_q/G_0 = G/G_0$ so that S_q intercepts every component of G and $S_q = G$. //

Examples and Special Cases.

E2.1 The Bundle of Linear Frames:

We consider here the bundle of linear frames $\pi_B : BM \to M$ over M, defined by

$$BM = \{p : \mathbb{R}^n \to T_xM \text{ , p linear isomorphism, } x \in M\} \text{ ,}$$

$n = \dim M$. BM admits the structure of a paracompact manifold of class C^{k-1} when M is of class C^k. With respect to this structure it is a principal bundle over M with structure group $Gl(n,\mathbb{R})$, which acts freely on BM by

$$R_a(p) = pa = p \circ a \text{ , } p \in BM \text{ , } a \in Gl(n,\mathbb{R}) \text{ .}$$

(c.f. [22], [26]). A large class of systems on BM satisfying (2.1) is obtained by lifting systems Σ on M :

Assume $k \geq 2$ so that the previous theorems are valid for systems evolving on BM. If ϕ is a local diffeomorphism of M, ϕ induces a local diffeomorphism $B\phi$ on BM by

$$B\phi(p) = \phi_* \circ p$$

that satisfies

$$\pi_B \circ B\phi = \phi \circ \pi_B \text{ and } R_a \circ B_\phi = B\phi \circ R_a \; ; \quad a \in Gl(n,\mathbb{R})$$

so that a vector field X defined on an open set $U \subset M$ lifts to a vector field BX on $\pi_B^{-1}(U)$ with one-parameter (pseudo) group $(BX)_t = B(X_t)$ and which satisfies condition (2.1).

This way, a system Σ on M gives rise to a system $B\Sigma$ on BM which projects onto Σ.

Since $B\Sigma$ is defined from Σ alone, all the objects introduced before concerning systems on principal bundles can be obtained, for $B\Sigma$, directly from Σ. For instance, $G_{B\Sigma}$ and $S_{B\Sigma}$ are the sets of local diffeomorphisms $B\phi$ with ϕ in G_Σ and S_Σ respectively.

Also, the group G_p, $p \in BM$, now a Lie subgroup of $Gl(n,\mathbb{R})$ is the group of matrices (w.r.t. the basis p) of the differentials of the local diffeomorphisms in G_Σ that fix $x = \pi_B(p)$. To see this, suppose that $a \in G_p$, then $pa = B\phi(p)$, i.e., $pa = (d\phi)_x \circ p$ for some $\phi \in G_\Sigma$ with $\phi(x) = x$. If (e_i) denotes the canonical basis in $\mathbb{R}^n$ and $a = (a^i_j)$, then $(d\phi)_x \circ p = pa$ is equivalent to

$$(d\phi)_x(pe_j) = p(\sum_i a_j^i e_i) = \sum_i a_j^i p(e_i) ,$$

so that the matrix of $(d\phi)_x$ w.r.t. the basis (pe_j) of T_xM is a.

To get S_p as a semi-group of matrices, just change G_Σ by S_Σ.

By the above, the Lie algebra $\mathfrak{g}_p$ of G_p becomes a Lie algebra of matrices, and as such we can view it the following way:

Under the identification of the fibre B_xM through p with $Gl(n,\mathbb{R})$ via $pa \to a$, the Lie algebra of G_p is identified with the intersection $T_p(B_xM) \cap T_p(G_{B\Sigma}(p))$ [proof: $\mathfrak{g}_p$ is the tangent space to G_p at the identity which is identified with the vertical part of the tangent space to $G_{B\Sigma}(p)$]. To see what this intersection is, put $B\Sigma_e = \{(B\phi)_*(BX) : B\phi \in G_{B\Sigma}$ and $BX \in B\Sigma\}$. Then $T_p(G_{B\Sigma}(p)) = $ $= \text{span}\{Z(p) : Z \in B\Sigma_e\}$, so that $T_p(B_xM) \cap T_p(G_{B\Sigma}(p))$ can be constructed by taking in the real vector space spanned by the germs at p of vector fields in $B\Sigma_e$ those that are vertical in p and evaluating them at p.

Now, the one-parameter group of $(B\phi)_*(BX)$ is $B\phi \circ (BX)_t \circ (B\phi)^{-1}$ which is easily seen to be equal to $B(\phi \circ X_t \circ \phi^{-1})$, but this is the one-parameter group of $B(\phi_*X)$, so that $(B\phi)_*(BX) = B(\phi_*X)$ and $B\Sigma_e = \{B(\phi_*X) : X \in \Sigma$ and $\phi \in G_\Sigma\}$. Therefore, if we put $\Sigma_e = \{\phi_*X : \phi \in G_\Sigma$ and $X \in \Sigma\}$ and denote by $\text{span}_x\Sigma_e$ the real vector space spanned by the germs at x of vector fields in Σ_e, we get that $\mathfrak{g}_p$ is identified with

$$\{BZ(p) : Z \in \text{span}_x\Sigma_e \text{ and } Z(x) = 0\} .$$

Taking local coordinates and p the canonical frame at x, one sees easily that $\mathfrak{g}_p$ is the Lie algebra of matrices given by the linear parts in the Taylor expansions of $Z \in \text{span}_x \Sigma_e$ with $Z(x) = 0$.

In the analytic situation, we do not need appeal to G_Σ to construct $\mathfrak{g}_p$. In fact, instead of Σ_e, we can take $LA(\Sigma)$ and the same description holds.

E2.2 : As a particular instance of the liftings above, take M to be a <u>Lie group</u> and the elements in Σ as <u>right invariant vector fields</u> in M. Then $G_p = \{1\}$ for every p. This is because the elements in G_Σ are left translations so that $\phi \in G_\Sigma$ fixes a point iff ϕ is the identity.

E2.3 : Generalizing E2.2, let M be a homogeneous space G/H with G a Lie group and H a closed subgroup. Take Σ to be a family of vector fields in M induced by a family of right invariant vector fields in G which generates the Lie algebra of G. Then $G_p \approx H/\bar{H}$ where $\bar{H}$ is the closed normal subgroup of H defined by

$$\bar{H} = \{h \in H : (dh)_x = 1\} \quad .$$

(In this expression x is the coset H in G/H and h is the map of M induced by $h \in H$) .

In case H is compact or semi-simple, $\bar{H} = \{1\}$. In fact, there is a coordinate system around x in which the elements of H are linear maps (c.f. [12] and [16]).

E2.4 Gradient Systems: Let M be a connected manifold of dimension n isometrically embedded in $\mathbb{R}^m$, $m > n$, and let

$$\Sigma = \{X_u : u \in \mathbb{R}^m\} \qquad (E2.4,1)$$

where X_u is the gradient of the linear function $x \mapsto \langle u,x\rangle$ restricted to M. The vector field X_u is also obtained by orthogonally projecting the constant vector field u on $\mathbb{R}^m$ into TM. If (e_i) is the canonical basis in $\mathbb{R}^m$, $u = \sum_{i=1}^{m} u_i e_i$ and Σ can also be described as

$$\Sigma = \{\sum_{i=1}^{m} u_i X_i : (u_1,\dots,u_m) \in \mathbb{R}^m\}$$

where $X_i = X_{e_i}$. This system is clearly transitive on M.

Let us give a lower bound for the structural group G_p of the lifting of Σ to BM. Denote by ∇ and ∇' the (Levi-Civita) connections of M and $\mathbb{R}^m$ respectively. Fix $x \in M$ and take an orthonormal basis $\xi_1,\dots,\xi_r$ $(r = m-n)$ of vectors orthogonal to M in a neighbourhood of x. Then if Y is a vector field in M defined around x,

$$\nabla'_Y X_u = \nabla'_Y(u - \sum_{i=1}^{m} \langle u,\xi_i\rangle\xi_i) = -\nabla'_Y(\langle u,\xi_i\rangle\xi_i)$$

$$= \sum_{i=1}^{m} Y(\langle u,\xi_i\rangle)\xi_i - \sum_{i=1}^{m} \langle u,\xi_i\rangle\nabla'_Y\xi_i \quad .$$

Hence the tangential component is given by

$$\nabla_Y X_u = \sum_{i=1}^{m} \langle u,\xi_i\rangle A_{\xi_i}(Y)$$

where A_{ξ_i} is the second fundamental form of M in the direction of ξ_i (c.f. [22] ch. VII), so that $\nabla_Y X_u = 0$ if $u \in T_x M$.

However, if $v \in T_x M$, X is any vector field and ω denotes the connection form, then

$$\nabla_v X(x) = \tfrac{1}{2}\, \omega_p(BX(p))(p^{-1}v)$$

for any $p \in \pi_B^{-1}(x)$ [to see this formula from the theory in [22], start with the equality $L_{BX}\theta = 0$ (prop. 2.1 ch. VI), where θ is the canonical form of BM , $\theta_p(w) = p^{-1}\, \pi_{B*}(w)$, $p \in BM$, $w \in T_p BM$. Denote by v^* the horizontal lift of v and write $L_{BX}\theta(v^*) = d i_{BX}\theta(v^*) + i_{BX} d\theta(v^*) = v^*.\theta(BX) + d\theta(BX,v^*)$. Now, use the first structure equation to see that $d\theta(BX,v^*) = -\tfrac{1}{2}\, \omega(BX)(v)$ and conclude that $v^*.\theta(BX) = \tfrac{1}{2}\omega(BX)(v)$, which gives the above formula by lemma 5.1 ch. III in [22]].

It follows that if u is tangent to M at x then $\omega(BX_u) = 0$, so that BX_u is horizontal on the fibre over x . This shows that the horizontal spaces of ∇ (in BM) are contained in the tangent spaces of the orbits of $B\Sigma$ and we conclude that if $p \in BM$ is an orthonormal framethen G_p contains the holonomy group of M .

As a consequence, we get that if $u,v \in T_xM$ then $R(u,v) \in \mathfrak{g}_p$ where $R(\cdot,\cdot)$ is the curvature form of M .

Now assume that M is a hypersurface in $\mathbb{R}^{n+1}$. For a unit normal field ξ , A_ξ is a symmetric transformation of T_xM . Let $f_1,\dots,f_n$ be an orthonormal basis of T_xM diagonalizing A_ξ : $A_\xi f_i = \lambda_i f_i$, $1 \le i \le n$. In this basis,

$$R(f_i,f_j) = \begin{array}{c|cc|} & i & j \\ i & 0 & \lambda_i\lambda_j \\ & & \\ j & -\lambda_i\lambda_j & 0 \end{array} .$$

In the special case of a compact hypersurface there is $x_0 \in M$ at which $\lambda_i \neq 0$ for all i (c.f. [22], ch. VII prop. 4.6) so that $SO(n,\mathbb{R}) \subset G_p$.

E2.5 Holonomy Bundles: As an example of a system $Q\Sigma$ that satisfies (2.2) but not (2.1) let us construct the holonomy bundles of a connection ω in a principal bundle Q (c.f. [22] ch. II, §7).

Take $Q\Sigma$ to be the set of all horizontal vector fields of ω : $QX \in Q\Sigma$ iff $\omega(QX) = 0$. If QX is horizontal, $R_{a*}QX$ might be different of QX , but $\omega(R_{a*}QX) = (R_a^*\omega)(QX) = Ad(a^{-1})\omega(QX) = 0$, i.e., $Q\Sigma$ satisfies (2.2) but not necessarily (2.1).

The horizontal spaces of ω are tangent to the orbits of $Q\Sigma$, thus the horizontal curves starting at $q \in Q$ stay in $G_{Q\Sigma}(q)$ and the holonomy bundles of ω are contained in the orbits of $Q\Sigma$.

Reciprocally, the trajectories of $Q\Sigma$ are horizontal curves so that $G_{Q\Sigma}(q)$ coincides with holonomy bundle passing through q. In this case, G_q is the holonomy group based at $q \in Q$.

E2.6 Standard Vector Fields: If in the above example we take $Q = BM$ then we can make the same reasoning to see that the holonomy bundles are the orbits of the system

$$St\Sigma = \{B(v) : v \in \mathbb{R}^n\} \tag{E2.6,1}$$

on BM, where $B(v)$ is a standard vector field on BM, i.e., $B(v)(p) \in T_pBM$ is the unique horizontal vector field at p that projects onto $pv \in T_xM$, $x = \pi_B(p)$ (see [22] ch.III).

Theorem 2.4 suggests the following refinement of the holonomy bundles:

Since $R_{a*}(B(v)) = B(a^{-1}v)$ (c.f. [22] prop. 2.2 ch. III), if $I \subset \mathbb{R}^n$ is invariant by the holonomy group then the system

$$St_I\Sigma = \{B(v) : v \in I\} \tag{E2.6,2}$$

on the holonomy bundle, satisfies (2.2) so that its orbits are principal bundles over submanifolds of M . Observe that if spI stands for the subspace of $\mathbb{R}^n$ spanned by I then spI is also invariant and $St_{spI}\Sigma$ is tangent to the orbits of $St_I\Sigma$ and since $St_I \subset St_{spI}$, these two systems have the same orbits.

E2.7 : In the above example take the linear connection ω to be invariant by parallelism (c.f. [22] ch. VII §7) and consider the lifting of $St\Sigma$ to the bundle of frames of BM as discussed in E2.1. Then the structure group of the orbits of this lifting is the identity.

Indeed, let e_i be a basis of $\mathbb{R}^n$ and A_j^* be a basis of fundamental vector fields in BM (the notations are as in [22]). Then the vector fields $(B(e_i),A_j^*)$ form a complete parallelism on BM , which by assumption has constant coefficients (i.e., the brackets of any two of these vector fields is a constant linear combination of the others). Therefore $B(e_i)$ preserves this parallelism and the orbit of the frame $(B(e_i)(p), A_j^*(p))$ $p \in BM$ is the set of frames $(B(e_i)(q),A_j^*(q))$ with q running over the holonomy bundle passing through p (c.f. [42] ch. I, th. 1.18).

E2.8 Trivial Bundles: $Q = M\times G$ with G a connected Lie group. The vector fields on Q satisfying (2.1) are of the type $QX(x,g) = (X(x),A(x,g))$ with X a vector field in M and $A(x,\cdot)$ a right invariant vector field in G . These vector fields are entirely determined by X and the map $A_{QX}(x) = A(x,1)$ from the domain of X into the Lie algebra $\mathfrak{g}$ of G .

The orbits of a system $Q\Sigma$ consisting of such vector fields are principal bundles over the orbits of $\Sigma = \pi_{Q*}(Q\Sigma)$, $\pi_Q(x,g) = x$. These orbits are not necessarily trivial bundles even in case Σ is transitive as is shown by example E2.9 below.

Let us see a situation in which the orbits of $Q\Sigma$ are trivial bundles.

Proposition E2.8.1 : Let $Q\Sigma$ be a system on $M \times G$ satisfying (2.1) and suppose that there exists a family Σ_{Hor} of vector fields on M such that the system

$$Q\Sigma_{Hor} = \{(X,0) : X \in \Sigma_{Hor}\}$$

is tangent to the orbits of $Q\Sigma$. Then Σ_{Hor} is tangent to the orbits of Σ and if its restriction to $G_\Sigma(x)$ is transitive then the orbits of $Q\Sigma$ over $G_\Sigma(x)$ are the trivial bundles

$$Ha \times G_\Sigma(x) \; , \quad a \in G$$

where H is the connected subgroup of G generated by $A_{QX}(x)$ with $QX \in Q\Sigma$.

Note: $Ha \times G_\Sigma(x)$ is a trivial $a^{-1}Ha$-principal bundle over $G_\Sigma(x)$.

Proof: By the assumptions, (x_1,g) and (x_2,g) are in the same orbit for all $x_1, x_2 \in G_\Sigma(x)$ and $g \in G$. The conclusion follows by the arguments in steps ii) and iii) in the proof of the proposition in the addendum to [3]. //

Systems satisfying the conditions of this proposition occur in [3] and [7] related to the study of linear systems driven by a real noise.

E2.9 : In E2.8 take $M = S^1$ and $G = S^1$. Then $M \times G$ is the two torus $\mathbb{R}^2/\mathbb{Z}^2$. In this bundle let $Q\Sigma$ be the system consisting of just one vector field, $Q\Sigma = \{\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial}{\partial y}\}$. Then the orbits of $Q\Sigma$ are double coverings of $M = S^1$ which are not trivial.

E2.10 <u>Equivariant Fibrations</u>: Let G be a Lie group (not necessarily connected) and $L_1 \subset L_2$ closed subgroups with L_1 normal in L_2. Then the map $\pi : gL_1 \in G/L_1 \to gL_2 \in G/L_2$, $g \in G$, which is equivariant by the actions of G on G/L_1 and G/L_2 (i.e. $\pi \circ g = g \circ \pi$) defines a principal bundle

$$\pi : G/L_1 \to G/L_2$$

with structure group L_2/L_1 (c.f. [37] §7.5). The action of L_2/L_1 on G/L_1 is defined by $(gL_1)(h_2L_1) = gh_2L_1$, $g \in G$, $h_2 \in L_2$ and commutes with the action of G on G/L_1.

If S is a semi-group in G that generates G and has non void interior then S induces a semi-group S_Q of diffeomorphisms of G/L_1 which satisfies conditions HS a) and c). In this case the semi-group

S_M is the semi-group of maps of G/L_2 induced by S via the action of G on G/L_2 . Also, if q is the coset L_1 in G/L_1 then $S_q = (S \cap L_2)/L_1$.

As an application of proposition 2.9, we have

Proposition E2.10.1 : Let G be a Lie group, L a closed subgroup of G with G/L compact and S a semi-group that generates G and has non void interior.

Suppose that there exists a finite sequence of closed subgroups

$$L = L_0 \subset L_1 \subset \ldots \subset L_k = G$$

such that L_j is normal in L_{j+1} .

Then S is controllable in G/L .

Proof: Since G/L is compact, G/L_j and L_{j+1}/L_j $j = 1,\ldots,k-1$; are compact. Hence by proposition 2.9, if S is controllable on G/L_{j+1} then S is controllable on G/L_j . But S is controllable on $G/L_k = \{1\}$ therefore on G/L . //

E2.11 : As a special case of E2.10, suppose that G is a Lie subgroup of $Gl(n,\mathbb{R})$ that acts transitively on $\mathbb{R}^n-\{0\}$. Then G also acts transitively on the sphere S^{n-1} and on the projective space $\mathbb{R}P^{n-1}$. If we put $\mathbb{R}P^{n-1} = G/L_1$, $S^{n-1} = G/L_2$ and $\mathbb{R}^n-\{0\} = G/L_3$ then L_1 and

L_2 are normal in L_3 and L_2/L_1 is compact. Hence by proposition 2.9 we have that a semi-group $S \subset G$, with $\operatorname{int} S \neq \emptyset$ and which generates G is controllable on S^{n-1} iff it is controllable on $\mathbb{R}P^{n-1}$.

Also, L_3/L_2 identifies with the multiplicative group of the positive reals and if $v \in \mathbb{R}^n-\{0\}$ then $S_v = \{\lambda \in \mathbb{R}^+ : gv = \lambda v$ for some $g \in S\}$. Therefore, if S is controllable on $\mathbb{R}P^{n-1}$ and if there are $g_1, g_2 \in S$ with $g_i v = \lambda_i v$, $i = 1,2$, and $\lambda_1 < 1$, $\lambda_2 > 1$ then S is controllable on $\mathbb{R}^n-\{0\}$.

§3. Invariant Control Sets on Fibre Bundles.

Given a principal bundle as in previous section we consider here systems evolving on associated fibre bundles $E(M,F,G,Q)$ constructed from $Q(M,G)$ by means of a left action of G on the typical fibre F (c.f. [22] Ch. I). The elements of E are viewed as equivalence classes for the equivalence relation $(q,v) \sim (q\,a, a^{-1}v)$, $Q \in G$, in $Q \times F$, and denoted by $q.v$, $q \in Q$, $v \in F$. With this notation, the canonical projection $\pi_E : E \to M$ is given by $\pi_E(q.v) = \pi_Q(q)$.

Let's recall the following maps: A fixed $q \in Q$ induces a bijection $v \in F \to q.v \in E_x$, $E_x = \pi_E^{-1}(x)$, $x = \pi_Q(q)$, between the fibre over x and the typical fibre. Thus q can be seen as a frame parameterizing E_x by F. A fixed $v \in F$ defines a map $v : Q \to E$ by $q \in Q \to q.v \in E$. In case the action of G on F is transitive this map is an onto submersion.

Now, let $Q\phi$ be an automorphism of Q which commutes with the right action of G : $Q\phi(qa) = Q\phi(q)a$. Then $Q\phi$ induces a fibre map $E\phi(q.v) = Q\phi(q).v$ on E. Therefore starting from an infinitesimal automorphism QX of Q we can construct - via its one parameter group - a vector field EX on E which projects down onto $X = \pi_{Q*}(QX)$ on M.

This way a system $Q\Sigma$ satisfying (2,1) induces a system $E\Sigma$ on E that projects onto a system Σ on M.

Each $E\phi$ in $G_{E\Sigma}$ (respectively $S_{E\Sigma}$) is given by a corresponding $Q\phi \in G_{Q\Sigma}$ (resp. $S_{Q\Sigma}$) so that for $q \in Q$ and $v \in F$ we have

$$\begin{aligned} &\text{a)} \quad G_{E\Sigma}(q.v) = G_{Q\Sigma}(q).v \\ &\text{b)} \quad S_{E\Sigma}(q.v) = S_{Q\Sigma}(q).v \end{aligned} \tag{3.1}$$

where for a set $A \subset Q$ we put $A.v = \{q'.v \in E : q' \in A\} =$ image under the map $v : Q \to E$ above of the set A.

Equalities (3.1) allow us to retrace the orbits and forward orbits of E from the corresponding orbits of $Q\Sigma$. For instance (3.1)a) together with theorem 2.4 implies that an orbit $G_{E\Sigma}(q.v)$ is a fibre bundle associated to the principal bundle $G_{Q\Sigma}(q)$ with base $G_\Sigma(x)$, $x = \pi_Q(q)$, and with the orbit of v in F under G_q as typical fibre. Thus settling a characterization of the orbits of $E\Sigma$.

The orbits of $E\Sigma$ being characterized this way, the study of its forward orbits is reduced to situation in which $Q\Sigma$ is transitive on Q and G is transitive on F.

In what follows we assume this situation in order to examine the invariant control sets of $E\Sigma$.

We deal here only with case when F is compact. But as in the previous section, the results are valid for semi-groups S_Q more general than those generated by families of vector fields. So we take

S_Q as in §2 and construct S_E acting on E as above. S_Q and S_E are related by (3.1)b).

Now, assume that S_Q satisfies HS. Then S_Q is accessible and transitivity of G on F implies accessibility of S_E. Also, from propositions 2.7 and 2.8 we have that for all $q \in Q$, S_q is a semi-group which has non void interior in G and generates G.

Therefore, fixing $q \in Q$ we have a finite number of S_q - i.c.s's on F. Denote them by C_q^j; $j = 1,\dots, ic(S_q,F)$; and define the sets $q.C_q^j \subset E_x$, $x = \pi_Q(q)$.

These sets are independent of the specific $q \in Q_x$.

In fact, let us take another q' in the same fibre as q. Then the semi-groups are related by $S_{q'} = a^{-1}S_q a$, from which it is readily seen that the $S_{q'}$ - i.c.s.'s are the sets $a^{-1}(C_q^j) \subset F$, $j = 1,\dots,ic(S_q,F)$. But $q'.a^{-1}(C_q^j) = q'a^{-1}.C_q^j = q.C_q^j$, so that the sets $q.C_q^j \subset E_x$ do not depend on q but only on x. We denote them by C_x^j. These are the building blocks for the invariant control sets of S_E in E.

<u>Theorem 3.1</u> : If S_Q satisfies HS, G is transitive on F and F is compact then

i) $ic(S_q,F)$ is constant as a function of $q \in Q$.

ii) There are S_E - i.c.s's and its number equals $ic(S_q,F)$.

iii) For every S_E - i.c.s. C and $q \in Q$,

$$q^{-1}(C \cap E_x), \quad x = \pi_Q(q) \tag{3.2}$$

is an invariant control set of S_q in F.

<u>Notes</u>: a) E is not assumed to be compact, so ii) is not automatic. b) It follows easily from iii) that if S_q is controllable for some q then $C = E$ and S_q is controllable for every q.

The proof requires some lemmas.

<u>Lemma 3.2</u> : Suppose that $Q\phi \in S_Q$ is such that $Q\phi(Qx) = Q_y$, $x,y \in M$. For $q_x \in Q_x$ let $q_y = Q\phi(q_x)$ and $C^j_{q_x}$ an S_{q_x} - i.c.s.

Thus for every $v \in C^j_{q_x}$,

$$C^j_{q_x} \subset \mathrm{cl}\, S_{q_y}(v) .$$

<u>Proof</u>: Pick arbitrary $v,w \in C^j_{q_x}$. We wish to find a sequence $(b_k)_{k\geq 1}$ in S_{q_y} with $b_k v \to w$.

By HS b) there exists $Q\psi \in S_Q$ such that $Q\psi \circ Q\phi(q_x) \in Q_x$. If we define $a \in G$ by $Q\psi \circ Q\phi(q_x) = q_x a$ then $a \in S_{q_x}$, so that $av \in C^j_{q_x}$. And since $w \in C^j_{q_x}$, there exists a sequence $(a_k)_{k\geq 1}$ in S_{q_x} with $a_k av \to w$.

Each a_k can be define by $Q\psi_k(q_x) = q_x a_k$, with $Q\psi_k \in S_Q$ and we have

$$Q\phi \circ Q\psi_k \circ Q\psi(q_y) = Q\phi \circ Q\psi_k \circ Q\psi \circ Q\phi(q_x)$$

$$= Q\phi \circ Q\psi_k(q_x a) = Q\phi(q_x a_k a) = q_y a_k a$$

so that $a_k a \in S_{q_y}$, which proves the lemma. //

Lemma 3.3 : With the notations as in lemma 3.2, $C^j_{q_x}$ is contained in a unique invariant control set of S_{q_y} . (Observe the non-symmetry of x and y .)

Proof: Let $v \in C^j_{q_x}$. By lemma 3.1 in [3] there exists a S_{q_y} - i.c.s. contained in $\mathrm{cl}\, S_{q_y}(v)$. Denote it by $C^j_{q_y}$. Since S_{q_y} has non void interior in G , $C^j_{q_y}$ has non void interior, hence $C^j_{q_y} \cap S_{q_y}(v) \neq \emptyset$ and there exists $b \in S_{q_y}$ with $bv \in C^j_{q_y}$.

We claim that there exists $\bar{b} \in S_{q_y}$ with $\bar{b}bv \in C^j_{q_x}$.

In fact, let $Q\psi' \in S_Q$ be such that $Q\psi'(q_y) = q_y b$ and take $Q\psi \in S_Q$ that maps Q_y into Q_x . Then

$$Q\phi \circ Q\psi(q_y) = q_y \bar{b}$$

with $\bar{b} \in S_{q_y}$. Also,

$$\begin{aligned} Q\phi \circ Q\psi \quad Q\psi'(q_y) &= q_y \bar{b}b = Q\phi(q_x)\bar{b}b \\ &= Q\phi(q_x\bar{b}b) \ , \end{aligned}$$

so that $Q\psi \circ Q\psi' \circ Q\phi(q_x) = q_x\bar{b}b$ and $\bar{b}b \in S_{q_x}$ and since $v \in C^j_{q_x}$, $\bar{b}bv \in C^j_{q_x}$, which proves the claim.

From the claim and the choice of b it follows that $\bar{b}bv \in C^j_{q_x} \cap C^j_{q_y}$, so by lemma 3.2 applied to $\bar{b}bv$, $C^j_{q_x} \subset C^j_{q_y}$. Uniqueness follows from the fact that the intersection of different S_{q_y} - i.c.s's is empty. //

In terms of the sets $C^j_{\mathbf{x}}$ this lemma interprets as follows:

Put $C^j_x = q_x.C^j_{q_x}$ and $C^j_y = q_y.C^j_{q_y}$ with $C^j_{q_y}$ as in the proof of lemma 3.3. If $E\phi$ corresponds to $Q\phi$ then $E\phi(C^j_x) = E\phi(q_x.C^j_{q_x}) = Q\phi(q_x).C^j_{q_x} \subset C^j_{q_y}$, thus $E\phi$ maps a set C^j_x into a unique $C^i_y = q_y.C^i_{q_y}$, $i = 1,\ldots, ic(S_{q_y})$. For the proof of theorem 3.1 we need to show that this $C^i_y \subset E_y$ is the same for every $Q\phi'$ that maps Q_x into Q_y.

<u>Lemma 3.4</u> : With $Q\phi$ as before take C^j_x and suppose that C^j_y is such that $E\phi(C^j_x) \subset C^j_y$. Then

a) If $Q\phi'$ maps Q_x into Q_y then $E\phi'(C^j_x) \subset C^j_y$.

b) If $Q\psi$ maps Q_y into Q_x then $E\psi(C^j_y) \subset C^j_x$.

<u>Proof</u>: Take $w \in C^j_{q_x}$. Then $E\psi(q_y.w) = Q\psi(q_y).w = Q\psi \circ Q\phi(q_x).w = q_x.aw$ for some $a \in S_{q_x}$, so that $E\psi(q_y.w) \in C^j_x$ and b) follows from lemma 3.3 applied to $Q\psi$.

To see a), write

$$Q\phi' \circ Q\psi(q_y) = q_y b$$

with $b \in S_{q_y}$. Then $E\phi'(E\psi(q_y.w)) = Q\phi' \circ Q\psi(q_y).w = q_y.bw \in C^j_y$, showing that $E\phi'(C^j_x) \cap C^j_y \neq \emptyset$, hence by lemma 3.3 $E\phi'(C^j_x) \subset C^j_y$. //

<u>Proof of theorem 3.1</u> : Lemma 3.4 implies that $ic(S_{q_x}) = ic(S_{q_y})$ so i) follows from controllability of S_M and the invariance of $ic(S_q)$ with q varying in a fixed fibre.

Now, pick $x \in M$ and some $C^j_x \subset E_x$. If $q.v \in C^j_x$ then by lemma 3.4 a) there is defined for each $y \in M$ a unique C^j_y with $C^j_y \cap S_E(q.v) \neq \emptyset$.

Put

$$C^j = \bigcup_{y \in M} C^j_y \ .$$

If $q'.w \in C^j$ then by lemma 3.4 b), $S_E(q'.v) \subset C^j$, and since C^j is in each fibre an invariant control set, controllability of S_M implies that $C^j \subset \mathrm{cl}\ S_E(q'.v)$. It follows that $\mathrm{cl}\ S_E(q'.w) = \mathrm{cl}\ C^j$, which by accessibility of S_E is sufficient to assure that C^j is an invariant control set.

This way we construct $ic(S_q)$ different S_E - i.c.s's, each of them satisfying (3.2).

To see that these are the only possibilities, let C be an S_E - i.c.s. and take $q.v \in C$. Then by the reasoning in the proof of lemma 3.3, $S_q v$ meets some S_q - i.c.s. in F . But $q.S_q v \subset S_E(q.v)$, which implies that for some C^j as above $S_E(q.v) \cap C^j \neq \emptyset$, i.e., $C \cap C^j \neq \emptyset$ so that $C = C^j$. This prove ii), iii) and the theorem. //

In the situation of the lemmas above, $E\phi(C_x^j)$ is in general properly contained in C_y^j . This happens - for instance - in case $E\phi(q.v) \in \mathrm{int}\ C^j$ for some q.v in the boundary of C^j .

However, if $(E\phi)^{-1}$ also belongs to S_E then lemma 3.4 shows that $E\phi(C_x^j) = C_y^j$. This means that if we parameterize E_x and E_y by q and $Q\phi(q)$ respectively then C_x^j and C_y^j are given by the same subset of F . In the next theorem we exploit this feature to show - for semi-groups generated by control systems - that if there are enough $E\phi \in S_E$ with $(E\phi)^{-1} \in S_E$ then the invariant control sets are locally trivial, i.e., are locally products of C_x^j with neighbourhoods of x in M (compare with example E2.8).

<u>Theorem 3.5</u> : Suppose that $Q\Sigma$ and G are transitive and F is compact. Consider the system $Q\Sigma \cap (-Q\Sigma)$. It is projectable onto a system $\Sigma_\pm$ on M which is symmetric: $X \in \Sigma_\pm$ iff $-X \in \Sigma_\pm$.

Let us assume that $\Sigma_\pm$ is transitive, or equivalently controllable.

Then, for every $x \in M$ there exists a local section $\sigma : U \subset M \to Q$, $x \in U$, such that the $E\Sigma$ - i.c.s's are given in $\pi_E^{-1}(U)$ by

$$\bigcup_{y \in U} \sigma(y) \cdot C^j_{\sigma(x)}$$

with $C^j_{\sigma(x)}$ an $S_{\sigma(x)}$ - i.c.s.

Proof: Let $x \in M$. By the assumptions there exists $D = (X^1, \ldots, X^k) \subset \Sigma_{\pm}$ such that $\rho_{D,x}$ has rank n (the dimension of M) at τ and $\rho_{D,x}(\tau) = x$.

Let $QD = (QX^1, \ldots, QX^k) \subset Q\Sigma \cap (-Q\Sigma)$ be a sequence that projects onto D, take $q \in Q_x$ and construct a local section $\sigma : U \subset M \to Q$ by the same procedure as lemma 2.1.

This section is given by $\sigma(y) = \rho_{QD,q}(\tau_y)$, with τ_y uniquely defined for $y \in U$ and $\tau_x = \tau$.

Put $Q\phi_y = \rho_{DQ,\tau_y}$. Then $Q\phi_y(q) = \rho_{DQ,\tau_y}(q) = \rho_{DQ,q}(\tau_y) = \sigma(y)$, hence $Q\phi_y(Q_x) = Q_y$ and $(Q\phi_y)^{-1} \in S_{Q\Sigma}$, so that for $y \in U$

$$C_y^j = E\phi_y(C_x^j) = E\phi_y(q.C_q^j) =$$
$$= Q\phi_y(q).C_q^j = \sigma(y).C_q^j \quad .$$

In particular, $\sigma(x).C^j_{\sigma(x)} = C_x^j = \sigma(x).C_q^j$, hence $C^j_{\sigma(x)} = C_q^j$ and

$$C_y^j = \sigma(y).C^j_{\sigma(x)} \quad . \quad //$$

We now wish to drop the controllability assumption on M and construct invariant control sets of S_E over invariant control sets of S_M. Many situations can be covered by the following proposition. We assume as before that G is transitive on F.

<u>Proposition 3.6</u> : Suppose that S_M is accessible, let C be an S_M - i.c.s. and assume the existence of $C_0 \subset C$ satisfying

i) C_0 is open in M .

ii) C_0 is invariant by S_M , i.e., $S_M(x) \subset C_0$ if $x \in C_0$.

iii) The restriction $S_{Q,0}$ of S_Q to $\pi_Q^{-1}(C_0)$ satisfies HS w.r.t. the bundle $\pi_Q^{-1}(C_0)$ over C_0 (the restriction is possible by ii)).

Let $C_0^j \subset \pi_E^{-1}(C_0)$, $j = 1,\dots,k$; be the invariant control sets of the restriction $S_{E,0}$ of S_E to $\pi_E^{-1}(C_0)$ as constructed in theorem 3.1.

Then the invariant control sets of S_E over C are the sets $\mathrm{cl}\, C_0^j \subset E$, $j = 1,\dots, k$.

<u>Notes</u>: (1) Accessibility of M implies that C is closed.

(2) By virtue of ii), C_0 is dense in C .

<u>Proof</u>: Since $\mathrm{cl}\, C_0^{j_1} \neq \mathrm{cl}\, C_0^{j_2}$ we need only to show that each $\mathrm{cl}\, C_0^j$ is a S_E - i.c.s.. Fix j and take $q.v \in \mathrm{cl}\, C_0^j$.

Assume that there exist $E\phi \in S_E$ and open V with $V \cap \mathrm{cl}\, C_0^j \neq \emptyset$ and such that $E\phi(q.v) \in V$. Then $(E\phi)^{-1}(V) \cap C_0^j \neq \emptyset$ so that there exists $\bar{q}.v \in C_0^j$ with $E\phi(\bar{q}.v) \notin C_0^j$. But this is impossible because C_0^j being a $S_{E,0}$, $E\phi(\bar{q}.v) \subset \mathrm{cl}\, C_0^j \cap \pi_E^{-1}(C_0)$, which equals to C_0^j . Consequently $S_E(q.v) \subset \mathrm{cl}\, C_0^j$ for arbitrary $q.v \in \mathrm{cl}\, C_0^j$.

Now, accessibility of S_M implies the existence of $E\phi \in S_E$ with $E\phi(q.v) \in \pi_E^{-1}(C_0) \cap \text{cl } C_0^j$. Then accessibility of $S_{E,0}$ entials that $C_0^j \subset \text{cl } S_E(q.v)$, which together with the above and the fact that $\text{cl } C_0^j$ is closed shows that $\text{cl } C_0^j$ is an invariant control set of S_E . //

Remark: It is easily seen that any S_E - i.c.s. is projected onto a S_M - i.c.s. so the above proposition covers all S_E - i.c.s's.

In case S_Q is generated by a control system $Q\Sigma$, we have

Proposition 3.7 : Suppose Σ is accessible and let C be a Σ - i.c.s. Then there exists $C_0 \subset C$ satisfying the conditions of proposition 3.6 if in iii) we take any orbit of $Q\Sigma|_{\pi_Q^{-1}(C_0)}$ instead of $\pi_Q^{-1}(C_0)$.

Moreover, $E\Sigma|_{\pi_E^{-1}(C_0)}$ is transitive iff $E\Sigma|_{\pi_E^{-1}(\text{int } C)}$ is transitive, and in this case the $E\Sigma$ - i.c.s's over C are described by proposition 3.6.

Proof: Take $x \in \text{int } C$ and $D \subset \Sigma$ such that the rank of $\rho_{D,x}$ at some τ is the dimension of M .

Let $y = \rho_{D,x}(\tau)$ and put $C_0 = \text{int } S_\Sigma(y)$.

Then i) and ii) of proposition 3.6 are readily verified. To check iii) we need to see only that $\Sigma|_{C_0}$ is controllable. But this is the case because $x = \rho^{\sim}_{D,y}(-\tau)$ so that $\operatorname{int} S_{-\Sigma}(y) \quad C \neq \emptyset$ and every point of C can be controlled into y . Therefore any $z_1 \in C_0$ can be controlled into any other $z_2 \in C_0$ by a trajectory of Σ which does not leave C_0 by ii).

If $E\Sigma\big|_{\pi_E^{-1}(C_0)}$ is transitive then the orbit by $E\Sigma\big|_{\pi_E^{-1}(\operatorname{int} C)}$ of any point in $\pi_E^{-1}(\operatorname{int} C)$ contains $\pi_E^{-1}(C_0)$, therefore $E\Sigma\big|_{\pi_E^{-1}(\operatorname{int} C)}$ is transitive.

Finally, if α_t is a trajectory of $E\Sigma$ backward in time joining $q_1.v$ and $q_2.v$, $q_1.v \in \pi_E^{-1}(C)$ then α_t never leaves $\pi_E^{-1}(C_0)$ because the trajectory obtained from α_t by reverting direction joins $q_2.v$ and $q_1.v$ forward in time, hence is contained in $\pi_E^{-1}(C_0)$. Therefore transitivity of $E\Sigma\big|_{\pi_E^{-1}(\operatorname{int} C)}$ implies transitivity of $E\Sigma\big|_{\pi_E^{-1}(C_0)}$. //

<u>Remarks</u>: (1) Transitivity of $E\Sigma\big|_{\pi_E^{-1}(\operatorname{int} C)}$ is equivalent to transitivity on F of the structural groups G_q of the orbits of $Q\Sigma\big|_{\pi_Q^{-1}(\operatorname{int} C)}$. If this condition fails, proposition 3.7 still works on the fibre bundles obtained by taking as typical fibres the compact G_q-orbits on F . In §5 we show that for certain G , G_q and F , the closed i.c.s's of semi-groups like the S_q's defined

before are always contained in a closed G_q-orbit on F, so that proposition 3.7 gives in this case the closed i.c.s's of $E\Sigma$ over C.

(2) $-\Sigma$ in example 1.2 is a case in which C_0 of proposition 3.7 is not $\operatorname{int} C$.

(3) C_0 in proposition 3.7 is the set of $z \in \operatorname{int} C$ such that $\operatorname{int} S_{-\Sigma}(z) \cap C \neq \emptyset$. In fact, if $z \in C_0$ then $\operatorname{int} S_{-\Sigma}(z) \cap C \supset \operatorname{int} S_{-\Sigma}(y) \cap C \neq \emptyset$, and if $\operatorname{int} S_{-\Sigma}(z) \cap C \neq \emptyset$ then $S_{-\Sigma}(z) \cap C_0 \neq \emptyset$ and $z \in C_0$ because C_0 is invariant by S_Σ. In particular, if Σ satisfies the Lie algebra rank condition at every $x \in C$, $C_0 = \operatorname{int} C$.

<u>Examples and Special Cases</u>.

E3.1 <u>Flags on Tensor Bundles</u>:

In case $Q = BM$ a system Σ on M lifts to a system $B\Sigma$ on BM as in E2.1 giving rise to systems $F\Sigma$ on any fibre bundle FM associated to BM via an action of $Gl(n,\mathbb{R})$ on the typical fibre F.

Examples of such F's are provided by subspaces V of tensors over $\mathbb{R}^n$ invariant by $Gl(n,\mathbb{R})$ (e.g. $\mathbb{R}^n$ or its dual $\mathbb{R}^{n*}$; $\Lambda^p\mathbb{R}^n$ = the p-th exterior product of $\mathbb{R}^n$). The bundles thus constructed are the tensor bundles. Other examples are provided by taking $F = \mathbb{F}(V;i_1,\dots,i_k)$, the space of all flags $\{V_1 \subset \dots \subset V_k\}$ of subspaces of V with $\dim V_j = i_j$, $j = 1,\dots,k$. The corresponding bundle $\mathbb{F}(V;i_1,\dots,i_k)$ is a bundle of flags of subspaces in some tensor bundle. For instance, $\mathbb{F}(\mathbb{R}^n;\, i_1,\dots,i_k)M$ is the bundle of flags of subspaces of TM.

In case Σ is transitive and controllable, theorem 3.1 gives the number of invariant control sets of $F\Sigma$ on the orbits with compact fibres, i.e., on the orbits $G_{F\Sigma}(p.v) = G_{B\Sigma}(p).v$, $p \in BM$, $v \in F$, such that $G_p(v)$ is a compact G_p-orbit.

The number of $F\Sigma$-i.c.s's is bounded above by the index of controllability of $G_p(v)$ as a G_p-homogeneous space. In §5 we show that in case G_p is semi-simple or reductive then the compact G_p-orbits on the above spaces of flags are unique controllable.

Systems and semi-groups on flag bundles have been considered in connection with the Lyapunov numbers of stochastic flows (c.f. [5], [9], [33] and reference therein).

E3.2 : For the <u>gradient systems</u> Σ of E2.4 it was shown that in case M is a hypersurface, G_p contains $SO(n,\mathbb{R})$ for some p , so that its lifting to any of the bundles $\mathbb{F}(\mathbb{R}^n, i_1,\ldots,i_k)M$ is transitive and since the system is symmetric $(X \in \Sigma \Rightarrow -X \in \Sigma)$, the only Σ-i.c.s. is the bundle itself.

Let us consider here for $B \subset \mathbb{R}^{n+1}$ the system

$$\Sigma_B = \{X_u : u \in B\} \tag{E3.2, 1}$$

with X_u as in E2.4.

Assume M is a hypersurface.

Then Σ_B is transitive iff B spans $\mathbb{R}^{n+1}$ and in this case Σ_B

is accessible. In fact, if span $B = \mathbb{R}^{n+1}$ then $\Sigma_B(x) = \{X_u(x) : u \in B\}$ spans T_xM for every $x \in M$, so that Σ_B satisfies the Lie algebra rank condition hence is transitive and accessible. Reciprocally if span $B = V \neq \mathbb{R}^{n+1}$ then for $x \in M$, $G_{B\Sigma}(x) \subset V$ and since M is a hypersurface, Σ_B is not transitive.

Assume from now on that span $B = \mathbb{R}^{n+1}$. Then any Σ_B-i.c.s. C is closed.

Let us verify that in case M is compact, $SO(n,\mathbb{R})$ is contained in the structural group of $\Sigma_B|_{\text{int } C}$.

Indeed, the orbits of $B\Sigma_B|_{\text{int } C}$ are the same as the orbits of $B\Sigma|_{\text{int } C}$ (because span $B = \mathbb{R}^n$) and since the proof in E2.5 that the holonomy group is contained in G_p was entirely local, it can be done inside int C.

Now, $SO(n,\mathbb{R})$ acts transitively on the flags $\mathbb{F}(\mathbb{R}^n;i_1,\ldots,i_k)$ so that G_p is also transitive hence the assumptions of proposition 3.7 are satisfied for the lifting of Σ_B to the flag bundles. In §5 it is shown that $\mathbb{F}(\mathbb{R}^n;i_1,\ldots,i_k)$ is unique controllable as a homogeneous space of a transitive linear group. Therefore the number of invariant control sets of the lifting of Σ_B to $\mathbb{F}(\mathbb{R}^n;i_1,\ldots,i_k)M$ is the same as the number of Σ_B-i.c.s's in M.

In case $M = S^n$, the Σ_B-i.c.s's are easily characterized: Let clco B denote the closed convex cone generated by B. Then Σ_B has a unique i.c.s. C which is given by

$$C = \text{clco } B \cap S^n .$$

To see this, recall that the closure of the forward orbits are not changed if instead of B we take the convex set it spans (c.f. Hermes and Lassalle [17], th.20.2), so that the closure of the forward orbits of Σ_B and $\Sigma_{\text{clco } B}$ are the same. Hence these two systems have the same invariant control sets.

Now, if $u \in \text{clco } B \cap S^n$ then $\lim_{t\to+\infty} (X_u)_t(x) = u$ for all $x \neq -u$, so that there is a unique $\Sigma_{\text{clco } B}$ - i.c.s. - denoted by C - which contains $\text{clco } B \cap S^n$. Since X_u points into clco B, $C \supset \text{clco } B \cap S^n$.

Observe that in this case the boundary of C is a well behaved subset of S^n: The boundary of any convex set is in a dense subset a C^2-submanifold.

E3.3 : Associated with the principal bundles of E2.10 there are the <u>equivariant fibrations of homogeneous spaces</u>:

Let G be a Lie group and $L_1 \subset L_2$ closed subgroups of G. Then the map $\pi_E : gL_1 \in G/L_1 \to gL_2 \in G/L_2$, $g \in G$ defines a fibre bundle

$$\pi_E : G/L_1 \to G/L_2$$

associated to any principal bundle $\pi_Q : G/\bar{L}_1 \to G/L_2$ with $\bar{L}_1 \subset L_1$ closed and normal in L_2 (c.f. [37] §7.5). The typical fibre of $G/L_1 \to G/L_2$ is L_2/L_1 and the left action of $L_2/\bar{L}_1$ on L_2/L_1 is given by $(h_2'\bar{L}_1).(h_2L_1) = h_2'h_2L_1$, $h_2', h_2 \in L_2$, which is readily seen to be well defined and transitive. Also, π_E is equivariant by the actions of G on G/L_1 and G/L_2, i.e., $g \circ \pi_E = \pi_E \circ g$.

If S is a semi-group in G with non void interior and which generates G, then S induces semi-groups S_Q on $G/\bar{L}_1$, S_E on G/L_1 and S_M on G/L_2. In the sequel all these semi-groups are denoted by S. The distinction is made by specifying the space in which S is acting.

In case S is controllable in G/L_2, it satisfies HS w.r.t. $G/L_1 \to G/L_2$ and when L_2/L_1 is compact, theorem 3.1 gives the invariant control sets of S on G/L_1. Its number is bounded by the index of controllability of L_2/L_1 as a homogeneous space of $L_2/\bar{L}_1$. This index of controllability is the same as $ic(L_2/L_1, L_2)$ because $\bar{L}_1 \subset L_1$ and is normal in L_2.

In general, in order to compare the S-i.c.s's on G/L_1 with the S-i.c.s's on G/L_2 we must check the conditions of proposition 3.6. Here we construct C_0. Further analysis are postponed to §6.

Proposition E3.3.1 : Let G be a Lie group, L a closed subgroup and S a semi-group with non void interior. Suppose that C is an S-i.c.s. on G/L.

Then there exists open $C_0 \subset C$ such that $C_0 = S(C_0) = \{gz : g \in S, z \in C_0\}$.

Proof: Take $x \in \text{int}\, C$ and $g \in \text{int}\, S$. Let $y = g(x)$ and put

$$C_0 = \text{int}(Sy) .$$

Then $(\text{int } S)^{-1}y \cap Sy \neq \emptyset$ so that $y \in (\text{int } S)Sy \subset (\text{int } S)y \subset C_0$ and $C_0 = (\text{int } S)y$. Therefore $C_0 \subset Sy \subset S(C_0) \subset (\text{int } S)y = C_0$. //

<u>Remarks</u>: (1) As in the case of control systems, C_0 is the set of $z \in \text{int } C$ such that $(\text{int } S)^{-1}z \cap C \neq \emptyset$ (compare with remark (3) following proposition 3.7). Therefore, $z \in C_0$ iff there exists $g \in \text{int } S$ with $gz = z$.

(2) It follows from (1) that when $C = G/L$, $C_0 = C$, i.e., S is controllable on G/L .

§4. Invariant Control Sets on the Boundaries of Semi-Simple Lie Groups.

In this section we will study the invariant control sets of semi-groups on compact homogeneous spaces of the type G/P, with G a semi-simple Lie group and P a parabolic subgroup of G. We consider only semi-groups with non void interior in G. The homogeneous spaces G/P are the Furstenberg boundaries of G and as will be seen afterwards, the invariant control sets on these spaces can be used as models for the invariant control sets on more general homogeneous spaces.

Let $\mathfrak{g}$ stand for the Lie algebra of G. We take $\mathfrak{g}$ to be a real non compact semi-simple Lie algebra. This is essentially the only requirement needed for our purposes. But we assume here that G is a connected Lie group (an alytic group) and show later (§6) how the non connected case can be reduced to this one.

When not specified on the contrary, the statements in this section will be valid for arbitrary connected semi-simple G.

However, to write proofs that involve the boundaries G/P we can assume further that G has finite centre or is even centreless.

This is because P being a parabolic subgroup, P is the normaliser in G of its Lie algebra $\mathfrak{p}$, which is a parabolic subalgebra (see Warner [45] or Varadarajan [44]). Thus if we make G act - via its adjoint representation on $\mathfrak{g}$ - on the Grassmannian $Gr_k(\mathfrak{g})$, $k = \dim \mathfrak{p}$, of k-dimensional subspaces of $\mathfrak{g}$, we see that P is the isotropy subgroup at $\mathfrak{p}$ so that G/P has a concrete realization as the orbit of $\mathfrak{p}$ under this action and the action of G on G/P depends only on the linear group $Ad(G) = \{Ad(g) : g \in G\}$ (Ad = adjoint representation of G), which for connected G equals the group $Inn(\mathfrak{g})$ of the inner automorphisms of $\mathfrak{g}$ and this group has finite centre: it is centreless.

This reduction of G will be used without any comment. Also, we will usually identify the elements of G/P with the conjugates of $\mathfrak{p}$ given by the realization of G/P mentioned above and with the conjugates of P provided by the fact that P is its own normalizer in G.

With these facts in mind, we can state

Theorem 4.1 : The boundaries G/P are unique controllable, that is, if S is a semi-group in G with $\text{int}\, S \neq \emptyset$ then S has a unique invariant control set on G/P.

Proof: If P' is a parabolic subgroup of G then P' contains a minimal parabolic subgroup P (see [45], pg. 55 or [44] part II ch. 6). We have thus an equivariant map

$$G/P \to G/P'$$

and in view of theorem 6.2 of §6, we can restrict ourselves to the case when P is a minimal parabolic subgroup.

So let $B = G/P$, with P minimal parabolic.

We will prove the uniqueness of S - i.c.s. by showing the existence of some $b_0 \in B$ with $b_0 \in \text{cl}\, Sb$ for every $b \in B$. If such a b_0 exists then S has a unique i.c.s. in B by lemma 3.1 in [3].

To find b_0 like this one we use the following lemma:

<u>Lemma 4.2</u> : There exists an Iwasawa decomposition $\underset{\sim}{g} = \underset{\sim}{k} + \underset{\sim}{a} + \underset{\sim}{n}^+$ of $\underset{\sim}{g}$ and $H \in \underset{\sim}{a}$ such that $h = \exp H \in \text{int}\, S$. Moreover H can be chosen to be $\underset{\sim}{a}$-regular in the sense that $\lambda(H) \neq 0$ if λ is a root of the pair $(\underset{\sim}{g},\underset{\sim}{a})$.

<u>Note</u>: By changing $\underset{\sim}{n}^+$ if necessary we can assume that H is in the positive Weyl chamber implicit in the Iwasawa decomposition, i.e., $\lambda(H) > 0$ if λ is a root in the positive system used to define $\underset{\sim}{n}^+$.

Before proving this lemma let us see how theorem 5.1 follows from it, so let us get b_0 with $b_0 \in \text{cl}\, Sb$ for all $b \in B$.

Let $G = KAN^+$, $K = \exp \underset{\sim}{k}$, $A = \exp \underset{\sim}{a}$, $N^+ = \exp \underset{\sim}{n}^+$, be the global decomposition of G corresponding to the Iwasawa decomposition of the lemma. If M denotes the centralizer of A in K then the subgroup $P_0 = MAN^+$, is minimal parabolic in G, and since all the minimal parabolic subgroups of G are conjugate we can view B as the coset space G/P_0.

Put $b_0 = P_0$ in this coset space.

Denote by $\underset{\sim}{n}^-$ the nilpotent subalgebra "opposed" to $\underset{\sim}{n}^+$. Thus if H is as in the lemma, $\underset{\sim}{n}^-$ is the sum of the eigenspaces of $ad(H)$ (ad = adjoint representation of $\underset{\sim}{g}$) associated with the negative eigenvalues. Hence if $N^- = \exp(\underset{\sim}{n}^-)$ then $\lim_{k\to+\infty} h^k n h^{-k} = 1$ for all $n \in N^-$.

By the Bruhat decomposition [14], [45], $N^- MAN^+ \subset G/MAN^+ = B$ is open and dense in B. Hence for any $b' \in B$ there exists $g \in S$ with $gb' \in N^- MAN^+$.

However, if $b = nMAN^+ \in N^- MAN^+$, with $n \in N^-$, then $h^k(b) = h^k n h^{-k} MAN^+$, so that $h^k(b) \to b_0$ as $k \to \infty$, and since $h^k \in \operatorname{int} S$, b_0 is as we wanted to be.

<u>Proof of lemma 4.2</u> : Since $\operatorname{int} S \neq \emptyset$, $\operatorname{int} S$ contains a regular element of G (c.f. [45] 1.3.4), therefore $\operatorname{int} S$ meets some Cartan subgroup of G ([45] 1.4.1.7). Denote this Cartan subgroup by J and let $\underset{\sim}{j}$ be its Lie algebra. J is the centralizer of $\underset{\sim}{j}$ in G which is an abelian subalgebra of $\underset{\sim}{g}$.

It is always possible to find a Cartan decomposition $\underset{\sim}{g} = \underset{\sim}{k} + \underset{\sim}{S}$ of $\underset{\sim}{g}$ such that $\underset{\sim}{j}$ decomposes as $\underset{\sim}{j} = \underset{\sim}{j} \cap \underset{\sim}{k} + \underset{\sim}{j} \cap \underset{\sim}{S} = \underset{\sim}{j}_{\underset{\sim}{k}} + \underset{\sim}{j}_{\underset{\sim}{S}}$ ([45] 1.3.1.1).

Let $K = \exp \underset{\sim}{k}$ and $J_K = K \cap J$. Then J_K a compact subgroup of G and J admits the decomposition $J = J_K \exp(\underset{\sim}{j}_{\underset{\sim}{S}})$ ([45] 1.4.1.2).

Define the subset σ of J_K by requiring that $u \in \sigma$ iff there exists $g \in \operatorname{int} S \cap J$ with $g = uh$ for some $h \in \exp(\underset{\sim}{j}_{\underset{\sim}{S}})$. Since in this decomposition u commutes with h, σ is a semi-group with non void interior in J_K. J_K being compact, σ contains the identity component of J_K, hence $\text{int } S \cap \exp(\mathfrak{j}_S) \neq \emptyset$.

Now, let $\mathfrak{a}$ be a maximal abelian subalgebra contained in $\mathfrak{S}$ and containing $\mathfrak{j}_S$. Then $\text{int } S \cap \exp(\mathfrak{a}) \neq \emptyset$, and since $\{H \in \mathfrak{a} : \lambda(H) \neq 0\}$ is open and dense the lemma follows. //

Taking G and S as in the theorem, denote by C the unique S-i.c.s. on the maximum boundary B of G and let $C_0 \subset \text{int } C$ be the set whose existence is ascertained in proposition E3.3.1. In what follows we shall characterize C_0 by means of the semi-simple elements in $\text{int } S$.

This characterization is done by relating elements in C_0 with semi-simple elements in $\text{int } S$ the same way as b_0 is related to h in the proof above. That is, b_0 is identified with the Lie algebra $\mathfrak{p}_0$ of P_0 which as a subspace of $\mathfrak{g}$ is the sum of $\text{Ad}(h)$-eigenspaces associated to the eigenvalues ≥ 1. Note that since $hb_0 = b_0$ and $h \in \text{int } S$, $b_0 \in C_0$ as follows from remark (1) after proposition E3.3.1.

In order to state precisely this characterization, we need some terminology. So fix a point $b_0 \in B$ and a Langlands decomposition $P_0 = MAN^+$ of its corresponding minimal parabolic subgroup (the isotropy at b_0). Associated with this decomposition there is an Iwasawa decomposition $G = KAN^+$ and the Weyl group W of the system of roots of the pair $(\mathfrak{g}, \mathfrak{a})$. The group W is a finite subgroup of the linear group of $\underset{\sim}{a}$. If M^* is the normalizer of $\underset{\sim}{a}$ in K then the action of W in $\underset{\sim}{a}$ is given by the adjoint action of M^* in $\underset{\sim}{a}$, hence $W = M^*/M$ (because M = centralizer of $\underset{\sim}{a}$ in K) and for every $w \in W$ there is a representative in $M^* \subset K$. In the sequel we always fix these representatives so when the Weyl group appears it is viewed as a subset of K . Because of this the action of W in $\underset{\sim}{a}$ is denoted, for $H \in \underset{\sim}{a}$, by $w(H)$ as well by $Ad(w)(H)$. Also, by looking at $w \in W$ as contained in K we have a well defined element $w(b_0) \in B$. It is given by the parabolic subgroup $wP_0w^{-1} = MA(wN^+w^{-1})$ or in terms of the coset space $B = G/P_0$ by wP_0 or yet by wM in the coset space $B = K/M$.

Suppose now that $\text{int } S \cap A \neq \emptyset$ and set

$$\Gamma = \{H \in \underset{\sim}{a} : \exp tH \in \text{int } S \text{ for some } t > 0\} .$$

Then Γ is a cone in $\underset{\sim}{a}$. Let us see that it is convex: Take $H \in \Gamma$. Then there exists an interval (a,b) , $b > a > 0$, such that $\exp tH \in \text{int } S$ if $t \in (a,b)$. Hence $\exp tH \in \text{int } S$ if $t \in (ka,kb)$ for positive integers k . But if k is large enough, $ka < (k-1)b$ therefore there exists $T > 0$ such that $\exp tH \in \text{int } S$ if $t > T$. Now, if $\exp t_1H_1$, $\exp t_2H_2 \in \text{int } S$ then (A is abelian)

$$\exp(t_1 + t_2)\left(\frac{t_1}{t_1+t_2} H_1 + \frac{t_2}{t_1+t_2} H_2\right) \in \text{int } S ,$$

hence $(t_1/t_1+t_2)H_1 + (t_2/t_1+t_2)H_2 \in \Gamma$. Making $t_1 \to +\infty$ by keeping t_2 fixed and vice-versa, we see that Γ is convex.

Define

$$\Lambda = \{h \in A : \exists\, n \in N^+ \text{ with } hn \in \operatorname{int} S\}$$

$$= \{h \in A : \exists\, n \in N^+ \text{ with } nh \in \operatorname{int} S\}$$

($hn = (hnh^{-1})h$ and A normalizer N^+) . Since $(h_1n_1)(h_2n_2) = h_1h_2\bar{n}$ (some $\bar{n} \in N^+$) , Λ is an open semi-group in A .

Define also

$$\tilde{\Lambda} = \{H \in \mathfrak{a} : \exp tH \in \Lambda \text{ for some } t > 0\} \quad .$$

Clearly $\Gamma \subset \tilde{\Lambda}$ and as Γ , $\tilde{\Lambda}$ is a convex cone in $\mathfrak{a}$.

From the proof of th. 4.1 we have that if $\mathfrak{a}^+$ stands for the positive Weyl chamber implicit in the decomposition $G = KAN^+$ then $b_0 \in C_0$ provided $\Gamma \cap \mathfrak{a}^+ \neq \emptyset$. The same thing happens in case $\tilde{\Lambda} \cap \mathfrak{a}^+ \neq \emptyset$ as can be seen by the note following the statement of theorem 4.4. below.

<u>Lemma 4.3</u> : Keep the above notations and let C be the unique S-i.c.s. in B . Suppose that $b_0 \in C_0$, $\operatorname{int} S \cap A \neq \emptyset$ and that $w(b_0) \in C_0$, with $w \in W$.

Then $w^{-1}(\Gamma) \subset \tilde{A}$.

<u>Proof</u>: Put $b^* = w(b_0)$ and $N^* = wN^+ w^{-1}$. The isotropy subgroup at b^* is $P^* = w P_0 w^{-1} = MAN^*$.

Since b_0 , $b^* \in C_0$ there are $g_1, g_2 \in \text{int } S$ with

$$g_1 b^* = b_0 \quad \text{and} \quad g_2 b_0 = b^*$$

(see proposition E3.3.1). We can write

$$g_1 = w^{-1} m_1 h_1 n_1 \tag{4.1}$$

for some $m_1 \in M$, $h_1 \in A$ and $n_1 \in N^*$. In fact, let $g_1 = u h_1 n_1$ be the decomposition of g_1 w.r.t. $G = KAN^*$. Then $g_1 b^* = (u h_1 n_1) b^* = ub^*$, so that $ub^* = b_0 = w^{-1} b^*$ and u belongs to the coset $w^{-1}M$ in the coset space $K/M = B$, i.e., $u = w^{-1} m_1$ for some $m_1 \in M$.

Similarly, we have

$$g_2 = w m_2' h_2' n_2' \tag{4.2}$$

with $m_2' \in M$, $h_2' \in A$ and $n_2' \in N^+$. We can rewrite (4.2) as $g_2 = (w m_2 w^{-1})w h_2' w^{-1})(w n_2' w^{-1})$ so that g_2 becomes

$$g_2 = m_2 h_2 n_2 w \tag{4.3}$$

for some $m_2 \in M$, $h_2 \in A$ and $n_2 \in N^*$.

Let $H \in \Gamma$. Then for $t > T$ some $T > 0$, $h_t = \exp tH \in \operatorname{int} S$, hence $g_1 h_t g_2 \in \operatorname{int} S$. However,

$$g_1 h_t g_2 = w^{-1} m_1 h_1 n_1 h_t m_2 h_2 n_2 w$$
$$= w^{-1} \bar{m} h_t h_1 h_2 \bar{n} w$$

with $\bar{n} \in N^*$ (because A normalizes N^* and $n_1, n_2 \in N^*$).
Putting $\bar{h}_t = w^{-1} h_t h_1 h_2 w$, we finally get

$$g_1 h_t g_2 = m_0 \bar{h}_t n_0 \in \operatorname{int} S$$

with $n_0 = w^{-1} \bar{n} w \in N^+$. Let us get rid of m_0 in this expression: Fix t and define the subset σ of M by requiring that $m \in \sigma$ iff $m \bar{h}_t^k n \in \operatorname{int} S$ for some positive integer k and $n \in N^+$. Then like in the proof of lemma 4.2 σ is a semi-group with non void interior in M and since M is compact, for each t there is an integer k such that $\bar{h}_t^k n \in \operatorname{int} S$ for some $n \in N^+$. This means that $\bar{H}_t = \log \bar{h}_t \in \Gamma$.

Now, keep h_1, h_2 fixed and make $t \to +\infty$. Then the ray defined by $\bar{H}_t$ approaches the ray defined by $w^{-1}(H)$. In fact, $\bar{H}_t = w^{-1}(\log h_t h_1 h_2)$ and the ray defined by $\log h_t h_1 h_2$ approaches the ray defined by H as $t \to +\infty$. It follows that $w^{-1}(H) \in \operatorname{cl} \tilde{A}$.

Since $H \in \Gamma$ was arbitrary, we have that $w^{-1}(\Gamma) \subset \mathrm{cl}\, \tilde{A}$.
But Γ and $\tilde{A}$ are open so that $w^{-1}(\Gamma) \subset \tilde{A}$. //

Theorem 4.4 : Let G and S be as before with $\mathrm{int}\, S \neq \emptyset$.
Denote by C the unique S-i.c.s. on the maximal boundary B of G and let C_0 be the set of controllability inside C as in proposition E3.3.1.

Then in order that $b \in B$ belongs to C_0 it is necessary and sufficient that there exists $g \in \mathrm{int}\, S$ satisfying:

i) $gb = b$

ii) If $P = MAN^+$ is some Langlands decomposition of the isotropy at b then

$$g = hn$$

with $n \in N^+$ and $h \in A^+ = \exp(\underset{\sim}{a}^+)$, the positive Weyl chamber in A .

Note: With $g = hn$ as in ii) it is always possible to find a decomposition $P = M_0 A_0 N_0^+$ with $g \in A_0^+$. In fact, take $n_0 \in N^+$ with $n_0\, hn\, n_0^{-1} = h$ and $M_0 = n_0^{-1} M n_0$, $A_0 = n_0^{-1} A n_0$, $N_0^+ = n_0^{-1} N^+ n_0 = N^+$.
The existence of such n_0 is ascertained in theorem 1.1.4.4 in [45].

Proof: Sufficiency follows from the proof of th. 4.1.

To prove that the condition is necessary, take $b \in C_0$. We will first find a convenient Langlands decomposition of the isotropy P at b, by showing that

(*) "There exists a Langlands decomposition $P = MAN^+$ such that $A \cap \text{int}\, S \neq \emptyset$."

Since $b \in C_0$ there exists $g \in \text{int}\, S$ with $gb = b$, hence $P \cap \text{int}\, S$ is a semi-group with non void interior in P.

Take some decomposition $P = M_0A_0N_0^+$ and set

$$\sigma = \{m \in M_0 : \exists\, g \in \text{int}\, S \text{ with } g = mhn\ ;\ h \in A_0\ ,\ n \in N_0^+\}\ .$$

Then σ is a semi-group with non-empty interior in the compact group M, so that $\text{int}\, S \cap A_0N_0^+ \neq \emptyset$.

Therefore, the argument in the note above gives a decomposition $P = MAN^+$ with $A \cap \text{int}\, S \neq \emptyset$, proving (*).

Keep this decomposition fixed and let $\mathfrak{a}^+$ be the positive Weyl chamber.

The union of the Weyl chambers being dense in $\mathfrak{a}$, there exists a chamber $\mathfrak{a}^*$ with $\mathfrak{a}^* \cap \Gamma \neq \emptyset$. Let $w \in W$ be the unique element in the Weyl group that satisfies $w(\mathfrak{a}^+) = \mathfrak{a}^*$ and put $N^* = w\, N^+\, w^{-1}$, $P^* = MAN^*$. Then $wPw^{-1} = P^*$ and if b^* corresponds to P^* then $w(b) = b^*$.

The chamber $\underset{\sim}{a}^*$ is the positive one for $P^* = MAN^*$ and since $\underset{\sim}{a}^* \cap \Gamma \neq \emptyset$, $b^* \in \text{int } C$, which in view of lemma 4.3 implies that $w^{-1}(\Gamma) \subset \tilde{\Lambda}$. But $w^{-1}(a^*) = a^+$ hence $\underset{\sim}{a}^+ \cap \tilde{\Lambda} \neq \emptyset$ and this is sufficient to prove the theorem. //

The maximum boundary $B = G/MAN^+$ is also the coset space K/M , and as such it can be interpreted as the set of Weyl chambers contained in the symmetric part of the decomposition $\underset{\sim}{g} = \underset{\sim}{k} + \underset{\sim}{s}$.

With B interpreted this way, the above theorem says that if a chamber b belongs to C_0 then modulo some nilpotent element int S intercepts this chamber.

This suggests that in case the chamber opposed to b is also in C_0 one would have int $S \cap$ int $S^{-1} \neq \emptyset$ and hence that $S = G$. In other words, the only possibility for S to be controllable in B , i.e., to have $C = B$, is when S is G itself. We prove next that this happens to be the case.

First let us see how to avoid the nilpotent elements alluded above. This is done via the following lemma that might be interesting in itself.

<u>Lemma 4.5</u> : Let G be a connected semi-simple Lie group with <u>finite centre</u> and S a semi-group in G . Suppose that there exists $X \in \underset{\sim}{g}$ with $\exp X \in \text{int } S$ and $\text{ad}(X)$ nilpotent.

Then $S = G$.

<u>Proof</u>: It is sufficient to consider the case of the group $\mathrm{Inn}(\mathfrak{g})$ of inner automorphisms of $\mathfrak{g}$: This group is the quotient $G/Z(G)$ of G by its centre. $Z(G)$ being finite, $G \to G/ZG)$ defines a principal bundle with compact group, so proposition 2.9 ii) applies.

To prove the lemma for this situation we approximate X by compact elements in $\mathrm{Inn}(\mathfrak{g})$.

Since $\mathrm{ad}(X)$ is nilpotent, the Jacobson-Morosov theorem says that X can be imbedded in a subalgebra $\mathfrak{g}_0$ isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ (see [45] 1.3.5.3 or [19] ch. III §11). Thus there exists $H, Y \in \mathfrak{g}$ such that

$$[H,X] = 2X \ ; \ [H,Y] = -2Y \ ; \ [X,Y] = H \ ,$$

and $\mathfrak{g}_0$ is the three dimensional Lie algebra generated by H, X and Y . Denote by G_0 the connected subgroup of $\mathrm{Inn}(\mathfrak{g})$ whose Lie algebra is $\mathfrak{g}_0$. G_0 is a semi-simple Lie subgroup of a linear group, hence G_0 has finite centre so that if $Z \in \mathfrak{g}_0$ then the one-parameter group $\{\exp tZ : t \in \mathbb{R}\} \subset G_0$ generated by Z in G_0 is compact iff $\{\exp t\, \mathrm{ad}_{\mathfrak{g}_0}(Z) : t \in \mathbb{R}\} \subset \mathrm{Inn}(\mathfrak{g}_0)$ is compact ($\mathrm{ad}_{\mathfrak{g}_0}$ meaning the adjoint inside $\mathfrak{g}_0$).

For $\varepsilon > 0$ take $Z_\varepsilon = X - \varepsilon^2 Y \in \mathfrak{g}_0$. Then the matrix of $\mathrm{ad}_{\mathfrak{g}_0}(Z_\varepsilon)$ w.r.t. the basis $\{X,H,Y\}$ is

$$\begin{vmatrix} 0 & -2 & 0 \\ \varepsilon^2 & 0 & 1 \\ 0 & -2\varepsilon^2 & 0 \end{vmatrix}$$

which has eigenvalues 0 and $\pm 2\varepsilon\sqrt{-1}$. Therefore $\{\exp t\, ad_{g_0}(Z_\varepsilon) : t \in \mathbb{R}\}$ is compact, hence $\{\exp t Z : t \in \mathbb{R}\}$ is compact in G_0 and consequently in $Inn(g)$.

But, if ε is small enough, $\exp Z_\varepsilon$ is near $\exp X$, so that $\exp Z_\varepsilon \in int\, S$. Hence $int\, S$ contains a neighbourhood of the identity in $Inn(g)$ and $S = Inn(g)$. //

Theorem 4.6 : Let G be a connected semi-simple Lie group with finite centre and S a semi-group in G with $int\, S \neq \emptyset$. Suppose that S is controllable on the maximum boundary B of G .

Then $S = G$.

Proof: In view of the lemma above we need only to find a nilpotent element in $int\, S$.

Clearly, S is controllable in B iff its i.c.s. is B itself.

Take $b \in B$ and let P be its corresponding parabolic subgroup. By th. 4.4 there exists a decomposition $P = MAN^+$ with $A \cap int\, S \neq \emptyset$ and by lemma 4.3, $w^{-1}(\Gamma) \subset \tilde{A}$ for every $w \in W$. Hence $\tilde{A}$ intercepts every chamber in a and since $\tilde{A}$ is a convex cone, $\tilde{A} = a$.

This means that there exists $h \in A$; $n_1, n_2 \in N^+$ such that hn_1 , $h^{-1}n_2 \in int\, S$, which implies that $N^+ \cap int\, S \neq \emptyset$ and the theorem follows from lemma 4.5. //

A special case: The group $Sl(n,\mathbb{R})$.

We will interpret here the above results for this group.

The boundaries of $Sl(n,\mathbb{R})$ are the flag manifolds:

Given a sequence of integers $k_1,\ldots,k_r$ with $1 \le k_1 \le \ldots \le k_r \le n$ we can form the set $\mathbb{F}^n(k_1,\ldots,k_r)$ of all flags $\{V_1 \subset \ldots \subset V_r\}$ of subspaces of $\mathbb{R}^n$ with $\dim V_j = k_j$, $j = 1,\ldots,r$.

The group $Sl(n,\mathbb{R})$ acts transitively on $\mathbb{F}^n(k_1,\ldots,k_r)$ by $g\{V_1 \subset \ldots \subset V_r\} = \{gV_1 \subset \ldots \subset gV_r\}$, $g \in Sl(n,\mathbb{R})$.

The sets $\mathbb{F}^n(k_1,\ldots,k_r)$ are the real flag manifolds and they are the boundaries of $Sl(n,\mathbb{R})$.

The maximal boundary is the flag manifold $\mathbb{F}^n(1,2,\ldots,n)$.

Given a flag $b = \{V_1 \subset V_2 \subset \ldots \subset V_n = \mathbb{R}^n\} \in \mathbb{F}^n(1,2,\ldots,n)$ the subgroup P of those $g \in Sl(n,\mathbb{R})$ for which $gb = b$ is the parabolic subgroup associated to b . By fixing a basis $\beta = \{e_1,\ldots,e_n\}$ contained in b , i.e., with $e_j \in V_j$, $j = 1,\ldots,n$, we get a Langlands decomposition $P = M_\beta A_\beta N_\beta^+$, where M_β is the set of $g \in Sl(n,\mathbb{R})$ whose matrix w.r.t. β is diagonal with entries ± 1 . A_β is the set of diagonal matrices with determinant one and positive entries while N_β^+ is the set of upper triangular matrices with 1's in the diagonal.

To take another basis contained in b amounts to make a conjugation of the decomposition by an element of N_β^+ as explained in the note following the statement of theorem 4.4.

Fix a flag b, a basis β contained in it and drop the subscript β in the above decomposition.

The positive Weyl chamber $A^+ = \exp(\underset{\sim}{a}^+)$ associated to this decomposition is the set of $h = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ with $\lambda_1 > \dots > \lambda_n > 0$. The other chambers are obtained from this one by applying a permutation on the entries of h. In fact, the Weyl group is the group of permutations

$$\mathrm{diag}(\lambda_1,\dots,\lambda_n) \to \mathrm{diag}(\lambda_{i_1},\dots,\lambda_{i_n}) .$$

If w is an element of the Weyl group, the flag $b^* = w(b)$ that appears in lemma 4.3 is the only flag that contains the basis $\beta^* = \{e_{i_1},\dots,e_{i_n}\}$ permutation of β.

Therefore lemma 4.3 means that if b and b^* are in C_0 and if there exists in int S an $h = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ then it is possible to find $\bar{h}\, n \in \mathrm{int}\, S$, with n upper triangular and $\bar{h} = \mathrm{diag}(\lambda_{j_1},\dots,\lambda_{j_n})$ where $(j_1,\dots,j_n)$ is the permutation inverse to $(i_1,\dots,i_n)$.

Theorem 4.4 interprets by saying that $b \in C_0$ iff there exists $g \in \mathrm{int}\, S$ such that w.r.t. some basis contained in b, $g = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ with $\lambda_1 > \dots > \lambda_n$. Its proof in this situation is as follows: If $b \in C_0$ then there exists $h = \mathrm{diag}(\lambda_1,\dots,\lambda_n) \in \mathrm{int}\, S$.

For some permutation $(i_1,\dots,i_n)$ we have $\lambda_{i_1} >\dots> \lambda_{i_n} > 0$, thus the chamber $\mathfrak{C}^*$ intercepts Γ and b^* constructed from $\beta^* = \{e_{i_1},\dots,e_{i_n}\}$ is also in C_0 (by theorem 4.1). Applying the permutation $(j_1,\dots,j_n)$ inverse of $(i_1,\dots,i_n)$ one sees that $\mathfrak{C}^*$ intercepts $\tilde{\Lambda}$.

In order to see another boundary of $Sl(n,\mathbb{R})$, let us consider the projective space $\mathbb{R}P^{n-1}$ which is the flag manifold $\mathbb{F}^n(1)$. The canonical projection $\pi : \mathbb{F}^n(1,2,\dots,n) \to \mathbb{R}P^{n-1}$ is given by $\pi\{V_1 \subset\dots\subset V_n\} = V_1$.

If S is a semi-group with non void interior in $Sl(n,\mathbb{R})$, its invariant control set $C_{\mathbb{P}}$ in $\mathbb{R}P^{n-1}$ is $\pi(C)$, where C is the S-i.c.s. on $\mathbb{F}^n(1,\dots,n)$.

We have thus from theorem 4.4 that $[v] \in \mathbb{R}P^{n-1}$ ($[v]$ = the class of $v \in \mathbb{R}^n - \{0\}$) belongs to $C_{\mathbb{P}}$ iff there exists $g \in \text{int}\, S$ with real eigenvalues $\lambda_1 >\dots> \lambda_n > 0$ such that $gv = \lambda_1 v$.

As a consequence, we can say that

"S is controllable in $\mathbb{R}^n - \{0\}$ iff S is controllable in $\mathbb{R}P^{n-1}$ ".

In fact, S is controllable in $\mathbb{R}P^{n-1}$ iff $C_{\mathbb{P}} = \mathbb{R}P^{n-1}$. Take $[v] \in \mathbb{R}P^{n-1}$ and $g \in \text{int}\, S$ with $gv = \lambda_1 v$ as above. Let $[w] \in \mathbb{R}P^{n-1}$ be such that $gw = \lambda_n w$. Since $g \in Sl(n,\mathbb{R})$, $\lambda_1 > 1 > \lambda_n$, and since $w \in C_{\mathbb{P}}$ there exists $g' \in \text{int } S$ with $g'w = \mu w$ for some $\mu > 1$. S is then controllable in $\mathbb{R}^n - \{0\}$ as in example E2.11.

In the next section it will be seen that the above controllability condition on $\mathbb{R}^n - \{0\}$ remains true if $Sl(n,\mathbb{R})$ is changed by any non compact semi-simple group that is transitive on $\mathbb{R}^n - \{0\}$.

Let us give an example of an invariant control set of a semi-group $S \subset Sl(n,\mathbb{R})$.

<u>Example</u>: Take $S \subset Sl(n,\mathbb{R})$ to be the semi-group of all matrices in $Sl(n,\mathbb{R})$ whose entries are ≥ 0 . This is a semi-group with non void interior and the identity is in the closure of the interior of S .

Then the unique invariant control set C of S in $\mathbb{R}P^{n-1}$ is the set corresponding to the positive orthant in $\mathbb{R}^n$, that is,

$$C = \{[(x_1,\ldots,x_n)] \in \mathbb{R}P^{n-1} : x_i \geq 0\} \quad .$$

In fact, the positive orthant is invariant by S so that C is contained in it. Also, $[e_1] = [(1,0,\ldots,0)] \in C$ because if $h = \text{diag}(\lambda_1,\ldots,\lambda_n)$ with $\lambda_1 > \ldots > \lambda_n > 0$ then $\lim_{k\to\infty} h^k[v] = e_1$ for $[v]$ in a dense subset.

Moreover, if $v = (x_1,\dots,x_n)$ with $x_1 > 0$ then $P[e_1] = [v]$ where

$$P = \begin{vmatrix} 1 & & & \\ x_2/x_1 & 1 & 0 & \\ \cdot & & & \\ \vdots & & & \\ x_n/x_1 & & & 1 \end{vmatrix},$$

which belongs to S . Therefore C contains the positive orthant.

Since $1 \in \mathrm{cl\ int}\ S$, the set C_0 is in this case $\mathrm{int}\ C$ itself.

§5. Subgroups.

Let $\bar{G}$ be a connected, non compact, semi-simple Lie group with Lie algebra $\bar{\mathfrak{g}}$ and $G \subset \bar{G}$ a Lie subgroup of $\bar{G}$ with the same properties and with Lie algebra $\mathfrak{g} \subset \bar{\mathfrak{g}}$.

We shall look here at the closed S-i.c.s's, $S \subset G$, int $S \neq \emptyset$, on the boundary manifolds of $\bar{G}$. What will be proved is that the closed i.c.s's are only those which can be found inside the closed orbits of G, and that these orbits are unique controllable as G-homogeneous spaces. As a result we get that the number of such closed i.c.s's is the same as the number of closed G-orbits.

As in the last section, we can assume that $\bar{G}$ and G are linear groups. Actually, for our purposes here we do not lose in generality if we assume that $\bar{G}$ is some $Sl(n,\mathbb{R})$. This is because any boundary of $\bar{G}$ can be imbedded as a closed $\bar{G}$-orbit in some Grassmannian of subspaces of $\bar{\mathfrak{g}}$, which in turn is a boundary manifold of the group $Sl(n,\mathbb{R})$, $n = \dim \bar{\mathfrak{g}}$. Clearly, the closed G-orbits and closed S-i.c.s's inside this $\bar{G}$-orbit can be considered as well as closed subsets of the Grassmannian. However, we do not bother to specify $\bar{G}$ until we need explicitly the structure of its parabolic subgroups, then it will be easier to take $\bar{G} = Sl(n,\mathbb{R})$.

Let us start by constructing compatible Iwasawa decompositions for G and G.

Suppose we are given some Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ ($\mathfrak{k}$ the subalgebra) of $\mathfrak{g}$ and a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$.

This decomposition can be extended to a compatible decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{s}}$, $\mathfrak{k} \subset \bar{\mathfrak{k}}$, $\mathfrak{s} \subset \bar{\mathfrak{s}}$ (c.f. [27] or [14] exercise A.8 ch. VI), so that we can take some maximal abelian $\bar{\mathfrak{a}} \subset \bar{\mathfrak{s}}$ with $\mathfrak{a} \subset \bar{\mathfrak{a}}$.

Let us denote by

$\bar{\pi}$ the set of roots of the pair $(\bar{\mathfrak{g}}, \bar{\mathfrak{a}})$.

Δ the set of roots of the pair $(\mathfrak{g}, \mathfrak{a})$.

π the set of restrictions $\bar{\lambda}|_{\mathfrak{a}}$ to $\mathfrak{a}$ of roots $\bar{\lambda} \in \bar{\pi}$.

$\bar{\Delta}$ the set of roots $\bar{\lambda} \in \bar{\pi}$ s.t. $\bar{\lambda}|_{\mathfrak{a}} \in \Delta$.

Then we have the decomposition

$$\bar{\mathfrak{g}} = \sum_{\bar{\lambda}\in\bar{\pi}} \bar{\mathfrak{g}}_{\bar{\lambda}} \tag{5.1}$$

of $\bar{\mathfrak{g}}$ in $\bar{\mathfrak{a}}$ -eigenspaces as well as the decomposition

$$\bar{\mathfrak{g}} = \sum_{\lambda\in\pi} \bar{\mathfrak{g}}_{\lambda} \tag{5.2}$$

in $\mathfrak{a}$-eigenspaces, where $\bar{\mathfrak{g}}_{\lambda} = \bar{\mathfrak{g}}_{\bar{\lambda}_1} \oplus \dots \oplus \bar{\mathfrak{g}}_{\bar{\lambda}_s}$ and $\bar{\lambda}_1, \dots, \bar{\lambda}_s$ are the roots in $\bar{\pi}$ that restricts to $\lambda \in \pi$. We also have the decomposition

$$\mathfrak{g} = \sum_{\lambda\in\Delta} \mathfrak{g}_{\lambda} \tag{5.3}$$

in $\mathfrak{a}$-eigenspaces, and since $\mathfrak{g}$ is a subalgebra, (5.3) is contained in (5.2), i.e., $\mathfrak{g}_\lambda \subset \bar{\mathfrak{g}}_\lambda$ and $\Delta \subset \pi$.

Now, choose a positive system of roots $\Delta^+ \subset \Delta$ and take the corresponding Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^+$, $\mathfrak{n}^+ = \sum_{\lambda \in \Delta^+} \mathfrak{g}_\lambda$. Let $\bar{\Delta}^+$ be the subset of $\bar{\Delta}$ consisting of those $\bar{\lambda} \in \bar{\Delta}$ whose restrictions are in Δ^+ . Then it is easily seen that $\bar{\Delta}^+ \cap (-\bar{\Delta}^+) = \emptyset$, so that $\bar{\Delta}^+$ extends to some positive system $\bar{\pi}^+ \subset \bar{\pi}$ of roots of $(\bar{\mathfrak{g}}, \bar{\mathfrak{a}})$.

It follows that

$$\mathfrak{n}^+ = \sum_{\lambda \in \Delta^+} \mathfrak{g}_\lambda \subset \sum_{\bar{\lambda} \in \bar{\Delta}^+} \bar{\mathfrak{g}}_{\bar{\lambda}} \quad \text{(because } \Delta \subset \pi)$$

$$\subset \sum_{\bar{\lambda} \in \bar{\pi}^+} \bar{\mathfrak{g}}_{\bar{\lambda}}$$

and hence:

<u>Lemma 5.1</u> : Every Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^+$ extends to an Iwasawa decomposition $\bar{\mathfrak{g}} = \bar{\mathfrak{k}} + \bar{\mathfrak{a}} + \bar{\mathfrak{n}}^+$; $\mathfrak{k} \subset \bar{\mathfrak{k}}$; $\mathfrak{a} \subset \bar{\mathfrak{a}}$, $\mathfrak{n}^+ \subset \bar{\mathfrak{n}}^+$. //

Clearly, these Iwasawa decompositions induce compatible global decompositions: $G = KAN^+$ and $\bar{G} = \bar{K}\bar{A}\bar{N}^+$, with $K \subset \bar{K}$, $A \subset \bar{A}$, $N^+ \subset \bar{N}^+$.

Fix these decompositions and take the minimal parabolic subgroup $\bar{P}_{min} = \bar{M}\bar{A}\bar{N}^+$ ($\bar{M}$ = centralizer of $\bar{A}$ in $\bar{K}$) . Let $\bar{P}$ be a parabolic subgroup of $\bar{G}$ containing $\bar{P}_{min}$. Then the G-orbit $G\bar{P}$ of $\bar{P}$ in the coset space $G/\bar{P}$ is compact.

In fact, for $g \in G$, write $g = uhn$ as its decomposition w.r.t. $G = KAN^+$. Then $hn \in \bar{P}_{min} \subset \bar{P}$ so that $g\bar{P} = u\bar{P}$ and G-orbit of $\bar{P}$ equals its K-orbit. Since we are assuming G to be linear, K is compact hence $G\bar{P}$ is compact.

We have thus a procedure to construct closed G-orbits in the boundary $G/\bar{P}$. As will be seen below (th. 5.4) every closed S-i.c.s. is contained in some of these G-orbits, in particular, every closed G-i.c.s., i.e., every closed G-orbit can be constructed this way.

Given the decomposition as above, let us introduce the following:

<u>Definition</u>: An element $H \in \mathfrak{a}$ is said to be regular w.r.t. $\bar{\mathfrak{g}}$ or $\bar{\mathfrak{g}}$-regular if

i) $\forall\ \lambda_1, \lambda_2 \in \pi$ with $\lambda_1 \neq \lambda_2$, $\lambda_1(H) \neq \lambda_2(H)$.

ii) $\forall\ \lambda \in \Delta$, $\lambda(H) \neq 0$.

<u>Note</u>: The set of $\bar{\mathfrak{g}}$-regular elements in $\mathfrak{a}$ is the non zero set of a finite number of linear maps in $\mathfrak{a}$. It is thus an open and dense subset of $\mathfrak{a}$.

Concerning these $\bar{\mathfrak{g}}$-regular elements, we have:

Lemma 5.2 : Let $H \in \underset{\sim}{a}$ be $\bar{\underset{\sim}{g}}$-regular and suppose that $V \subset \bar{\underset{\sim}{g}}$ is invariant by $ad(H) : \bar{\underset{\sim}{g}} \to \bar{\underset{\sim}{g}}$.

Then V is also invariant by $ad(H')$, $H' \in \underset{\sim}{a}$.

Proof: For $v \in V$ write $v = \sum_{\lambda \in \pi} v_\lambda$, with $v_\lambda \in \bar{\underset{\sim}{g}}_\lambda$. Then $ad(H)(v) = \sum_{\lambda \in \pi} \lambda(H) v_\lambda$ is also in V , and since H is $\bar{\underset{\sim}{g}}$-regular, all the coefficients $\lambda(H)$ in this sum are different, so we can take convenient linear combinations to see that each $v_\lambda \in V$. This means that $ad(H')(v) = \sum_{\lambda \in \pi} \lambda(H') v_\lambda \in V$, for arbitrary $H' \in \underset{\sim}{a}$. //

We will need the following lemma about the action of a linear map on Grassmannians.

Lemma 5.3 : Let H be a $m \times m$ diagonal matrix (real eigenvalues) and let $\exp tH$ act on the Grassmannian $Gr_k(m)$ of k-planes in $\mathbb{R}^m$.

Then for every $b \in Gr_k(m)$, $\lim_{t \to +\infty} e^{tH}(b)$ exists and is a k-plane invariant by H .

Proof: If the Grassmannian is a projective space, write $b = b_{max} + b_{others}$, where b_{max} is the component of b in the direction of the eigenspace of H associated with the highest eigenvalue and b_{others} = component w.r.t. the others eigenspaces. Then $\lim_{t \to +\infty} e^{tH}(b) = b_{max}$ and b_{max} is H-invariant.

For $Gr_k(m)$, look at it as a (closed) submanifold in the projective space of the k-fold exterior algebra of $\mathbb{R}^m$ and apply the above argument. //

Theorem 5.4 : Let $G \subset \bar{G}$ be as before and $S \subset G$ a semi-group with non void interior in G.

Then the closed S-i.c.s's on the boundary manifolds $\bar{G}/P$, P parabolic, of $\bar{G}$ are contained in the closed G-orbits.

Proof: The boundary $\bar{G}/\bar{P}$ will be viewed as the (closed) $\bar{G}$-orbit of the Lie algebra of $\bar{P}$ in the corresponding Grassmannian of subspaces of $\bar{\mathfrak{g}}$.

Let C be a S-i.c.s. in $\bar{G}/\bar{P}$ and pick $\bar{\mathfrak{p}}_0 \in C$. $\bar{\mathfrak{p}}_0$ will be moved within the closure $\mathrm{cl}\, C$ of C, until a closed G-orbit of the type constructed above is reached.

Take the Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}^+$ and $H \in \mathfrak{a}$ with $h = \exp H \in \mathrm{int}\, S$ as ascertained by lemma 4.2. We can assume H to be $\bar{\mathfrak{g}}$-regular.

From lemmas 5.2 and 5.3 we have that $\lim\limits_{j\to+\infty} h^j(\bar{\mathfrak{p}}_0) = \bar{\mathfrak{p}}_1$ (j integer) is a parabolic subalgebra invariant by $\mathrm{ad}(H')$, $H' \in \mathfrak{a}$, which in view of the fact that $\bar{\mathfrak{p}}_1$ is its own normalizer implies that $\mathfrak{a} \subset \bar{\mathfrak{p}}_1$. Clearly, $\bar{\mathfrak{p}}_1 \in \mathrm{cl}\, C$.

Define $\Delta^+ = \{\lambda \in \Delta : \lambda(H) > 0\}$. By the choice of H, Δ^+ is a positive system of roots of $(\mathfrak{g},\mathfrak{a})$. Let $\mathfrak{n}^+ = \sum\limits_{\lambda\in\Delta^+} \mathfrak{g}_\lambda$ be the corresponding nilpotent subalgebra.

Although $\underset{\sim}{a} \subset \bar{\underset{\sim}{p}}_1$ it is not necessarily true that $\underset{\sim}{n}^+$ is contained in $\bar{\underset{\sim}{p}}_1$, so we move $\bar{\underset{\sim}{p}}_1$ into another subalgebra that contains $\underset{\sim}{n}^+$.

In order to do that, let us take a basis $\{X_1,\ldots,X_r\}$ of $\underset{\sim}{n}^+$ formed by $\underset{\sim}{a}$-eigenvectors and such that the subspace span $\{X_1,\ldots,X_j\}$ is an ideal in span$\{X_1,\ldots,X_{j+1}\}$, $j = 1,\ldots,r-1$. It is not difficult to see from the structure of $\underset{\sim}{a} + \underset{\sim}{n}^+$ that a basis like this in fact exists.

In this basis, choose X_1 small enough in order that

$$h \exp(\mathrm{Ad}(h^{-1})X_1) \in \mathrm{int}\, S \ .$$

Then

$$\begin{aligned} e^{X_1} \bar{\underset{\sim}{p}}_1 &= h \exp(\mathrm{Ad}(h^{-1})X_1)h^{-1} \bar{\underset{\sim}{p}}_1 \\ &= h \exp(\mathrm{Ad}(h^{-1})X_1) \bar{\underset{\sim}{p}}_1 \in \mathrm{cl}\, C \end{aligned}$$

because $\bar{\underset{\sim}{p}}_1 \in \mathrm{cl}\, C$. This implies that $h^j e^{X_1} \bar{\underset{\sim}{p}}_1 \in \mathrm{cl}\, C$, for positive integers j .

Now, define - with the aid of lemma 5.3 - the parabolic subalgebra.

$$\bar{\underset{\sim}{p}}_2 = \lim_{\substack{t\to+\infty \\ t\in\mathbb{R}}} e^{tH} e^{X_1} \bar{\underset{\sim}{p}}_1 = \lim_{j\to+\infty} h^j e^{X_1} \bar{\underset{\sim}{p}}_1 \ .$$

Then $\bar{\mathfrak{p}}_2 \in \mathrm{cl}\, C$ and is H-invariant, which together with the fact that H is $\bar{\mathfrak{g}}$-regular implies that $\mathfrak{a} \subset \bar{\mathfrak{p}}_2$.

Let us show that $X_1 \in \bar{\mathfrak{p}}_2$.

If $\lambda \in \Lambda^+$ is such that $X_1 \in \mathfrak{g}_\lambda$ then $\lambda(H) > 0$ and

$$e^{tH} e^{X_1} \bar{\mathfrak{p}}_1 = e^{tH} e^{X_1} e^{-tH} \bar{\mathfrak{p}}_1$$

$$= \exp(\mathrm{Ad}(e^{tH})X_1)\, \bar{\mathfrak{p}}_1$$

$$= \exp(e^{t\lambda(H)}X_1)\, \bar{\mathfrak{p}}_1 \quad .$$

It follows that

$$\lim_{s\to+\infty} e^{sX_1} \bar{\mathfrak{p}}_1 = \lim_{t\to+\infty} \exp(e^{t\lambda(H)}X_1)\bar{\mathfrak{p}}_1$$

$$= \bar{\mathfrak{p}}_2 \quad .$$

Therefore, if we write $e^{sX_1} = e^{s_1X_1} e^{(s-s_1)X_1}$, $s_1 > 0$, we see that $e^{s_1X_1} \bar{\mathfrak{p}}_2 = \bar{\mathfrak{p}}_2$, $\forall\, s_1 > 0$, i.e., $\bar{\mathfrak{p}}_2$ is X_1-invariant and $X_1 \in \bar{\mathfrak{p}}_2$.

Applying the same arguments to X_2 , we get

$$\bar{\mathfrak{p}}_3 = \lim_{s\to+\infty} e^{sX_2} \bar{\mathfrak{p}}_2$$

with $\bar{\mathfrak{p}}_3 \in \mathrm{cl}\, C$ and invariant by $\mathfrak{a}$ and X_2 . It is also invariant by

X_1 because $X_1 \in \bar{p}_2$ and X_2 normalizes span$\{X_1\}$ so that $X_1 \in \mathrm{Ad}(e^{sX_2})\bar{p}_2 = e^{sX_2}\bar{p}_2$ for every $s \in \mathbb{R}$. Passing to the limit we see that X_1 in fact belongs to $\bar{p}_3$ (the set of subspaces that contain a fixed vector is closed in any Grassmannian).

Proceeding in this way we finally end in a parabolic subalgebra $\bar{p}_{r+1} \in \mathrm{cl}\ C$ that contains $a + n^+$ and consequently has a closed orbit in $\bar{G}/\bar{P}$.

Therefore cl C meets some closed G-orbit so if $\mathrm{cl}\ C = C$, C must be contained in this closed orbit. //

We wish now to examine the index of controllability of the closed G-orbits in G/P. As mentioned already we can always think of G as a linear group acting on some Grassmannian and take $\bar{G}$ to be $Sl(n,\mathbb{R})$.

Let us do that and look at the closed G-orbits on the maximal boundary manifold of $\bar{G} = Sl(n,\mathbb{R})$. The general case will be derived from this one.

Suppose that such a closed orbit is given as a G-homogeneous space by G/L and let L_0 be the identity component of L and ℓ its Lie algebra. It comes from the proof of the above theorem that there exist compatible decompositions $G = KAN$ and $\bar{G} = \bar{K}\bar{A}\bar{N}$ such that G/L is the orbit of $\bar{M}\bar{A}\bar{N}$ in $\bar{G}/\bar{M}\bar{A}\bar{N}$ ($\bar{M}$ = centralizer of $\bar{A}$ in $\bar{K}$). Hence modulo some conjugate $L \subset \bar{M}\bar{A}\bar{N}$ and since the identity component of $\bar{M}\bar{A}\bar{N}$ is $\bar{A}\bar{N}$

(because $\bar{G} = Sl(n,\mathbb{R})$) , we have that $L_0 \subset \bar{A}\bar{N}$. Also, $AN \subset \bar{A}\bar{N}$ so AN fixes $\bar{M}\bar{A}\bar{N}$ hence $AN \subset L$. AN being connected, $AN \subset L_0$.

It follows that $\underset{\sim}{a} + \underset{\sim}{n} \subset \underset{\sim}{\ell} \subset \bar{\underset{\sim}{a}} + \bar{\underset{\sim}{n}}$ (with obvious notation), hence $\underset{\sim}{\ell}$ is solvable and contains $\underset{\sim}{a} + \underset{\sim}{n}$. But $\underset{\sim}{a} + \underset{\sim}{n}$ is maximal solvable in $\underset{\sim}{g}$ therefore $\underset{\sim}{\ell} = \underset{\sim}{a} + \underset{\sim}{n}$, $L_0 = AN$ and L is contained in the normalizer of AN . This normalizer being the parabolic subgroup MAN , we get for L the characterization $L = (L \cap M)AN \subset MAN$.

Let us look in more detail at the compact group $L \cap M$.

Denote by M_0 the identity component of M and by $Z_{\mathbb{C}}$ the subgroup of $Sl(n,\mathbb{C})$ formed by elements of the type $\exp(\sqrt{-1}\,H)$, $H \in \underset{\sim}{a}$. Then if Z is the set of real elements in $Z_{\mathbb{C}}$, $Z \cap K$ abelian subgroup of K isomorphic to the group of components of M , i.e., $M = ZM_0$ (see [45] lemma 1.1.38, where a proof of this fact is made for the adjoint representation, but which is valid for an arbitrary representation thus covering our situation).

However, in the basis of $\mathbb{R}^n$ that diagolizes $\bar{A}$, the elements of $\underset{\sim}{a}$ are diagonal and so are the elements in Z . This means that $Z \subset \bar{M}\bar{A}\bar{N}$ hence $Z \cap K \subset L$, L intercepts the connected components of M and $M/M \cap L$ is connected.

Consider now the fibration

$$G/L \to G/MAN .$$

Since $AN \subset L$ and is normal in MAN , this is a fibre bundle with group $MAN/AN = M$ and typical fibre $M/M \cap L$. This fibre being connected, over any i.c.s. in G/MAN there are at most $ic(M/M \cap L)$ i.c.s's in G/L (see th. 6.2). But G/MAN is the maximal boundary of G so is unique controllable. Also, M is compact, so $M/M \cap L$ is not only unique controllable but even controllable.

Therefore G/L is unique controllable and the i.c.s's in G/L are of the type $\pi^{-1}(C)$ with C the corresponding i.c.s. in G/MAN .

In general, we have

<u>Theorem 5.5</u> : Let $G \subset \bar{G}$ be as before. Then the closed orbits of G on the boundaries of $\bar{G}$ are unique controllable.

<u>Proof</u>: If G/L_1 is one of these orbits then it is a closed G-orbit in some Grassmannian, hence by the characterization of the closed orbits given before we can find some closed G-orbit G/L in the maximal boundary of $Sl(n,\mathbb{R})$ such that $L_1 \supset L$. This defines a covariant fibration

$$G/L \to G/L_1 .$$

Since G/L is unique controllable, G/L_1 is also unique controllable. //

<u>Remarks:</u> (1) The previous theorems stated for semi-simple Lie groups are quickly extended to the case in which $\mathfrak{g}$ is reductive: Taking $\mathfrak{g}$ to be a linear Lie algebra, write $\mathfrak{g} = \mathfrak{z} + [\mathfrak{g},\mathfrak{g}]$ with $\mathfrak{z}$ the centre

and $[\mathfrak{g},\mathfrak{g}]$ semi-simple. If $\mathfrak{a} \subset [\mathfrak{g},\mathfrak{g}]$ denotes an abelian subspace of the kind described before, then in some ordering of the basis that diagonalizes $\mathfrak{a}$, the elements of $\mathfrak{z}$ are written as $\mathrm{diag}(z_1,\bar{z}_1,\ldots,z_s,\bar{z}_s,\lambda_1,\ldots,\lambda_k)$, with $z_j \in \mathbb{C}$, $\lambda_j \in \mathbb{R}$ and the bar denoting complex conjugation (c.f. Schur's lemma). So if we increase in the above proofs $\mathfrak{a}$ by the real elements in $\mathfrak{z}$ and $\mathfrak{m}$ by the purely imaginary ones (which generate a torus), we will get theorems 5.4 and 55 for this slightly more general situation.

(2) In case $\tilde{G}$ above is a complex Lie group and G a real form, the orbits of G on the boundaries of $\tilde{G}$ where studied by Wolf [46]. One of the results in [46] (c.f. theorem 3.3) is that there exists a unique closed G-orbit on the boundaries of $\tilde{G}$.

Now, take $G \subset Sl(n,\mathbb{R})$ semi-simple connected non compact and $S \subset G$ a semi-group with $\mathrm{int}\, S \neq \emptyset$. Let C be the unique S-i.c.s. in a closed G-orbit on a flag manifold (a boundary of $Sl(n,\mathbb{R})$).

Then as in theorem 5.5 above, C is the projection of the unique S-i.c.s. on G/L, a closed G-orbit on the maximum flag manifold. But the S-i.c.s. on G/L is the inverse image by the fibration $\pi: G/L \to G/MAN$ of the S-i.c.s. on the maximal boundary G/MAN of G. Hence by theorem 4.4, if $b \in C_0$ then b is a fixed point of some $h = \mathrm{diag}(\lambda_1,\ldots,\lambda_n) \in \mathrm{int}\, S$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0$. Since G is non compact, h can be chosen in such a way that $\lambda_1 > 1 > \lambda_m$.

Therefore, like in the case in which $G = Sl(n,\mathbb{R})$, we have

Proposition 5.6 : Let G be linear semi-simple connected and non-compact. Suppose that G is transitive on $\mathbb{R}^n - \{0\}$ and let S be a semi-group with $\operatorname{int} S \neq \emptyset$.

Then S is controllable in $\mathbb{R}^n - \{0\}$ iff S is controllable in $\mathbb{RP}^{n-1}$. //

In case G is not semi-simple this proposition is not true:

Example: Take $\Sigma = \{X,Y\}$ where X and Y are linear vector fields defined by the matrices

$$X = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \quad ; \quad Y = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}$$

Then the Lie algebra generated by Σ is $\underset{\sim}{gl}(2,\mathbb{R})$ so the semi-group S_Σ it generates has non void interior in $Gl^+(2,\mathbb{R})$. Σ is clearly controllable in $\mathbb{RP}^{n-1}$. However Σ is not controllable in $\mathbb{R}^n - \{0\}$ because Y is tangent to every circle while X points outwards the circles.

§6. On the Index of Controllability of Compact Homogeneous Spaces.

In this section we shall derive some results concerning the index of controllability of a homogeneous space $M = G/L$. The main point in these results is to give upper bounds to $ic(M,G)$ by using theorem 3.1, some fibration of M and the unique controllability of the spaces in §5. As a particular instance of what is done here, it will be proved that $ic(M,G)$ is always finite.

We start by reducing the problem of computing $ic(M,G)$ to the case when G is connected. In the sequel, the identity component of a Lie group J is denoted by J_0 .

Proposition 6.1 : Let G be a Lie group and L a closed subgroup with G/L compact. Note that $G_0 \cap L$ and G_0L are closed subgroups, $G_0/L \cap G_0$ being a connected component of G/L is compact and G/G_0L is the set of components of G/L . We have,

$$ic(G/L) = ic(G_0/L \cap G_0) .$$

Proof: Let S be a semi-group which generates G and has non void interior.

Consider the fibre bundle

$$G/L \to G/G_0L$$

over the set of components of G/L with fibre $G_0L/L = G_0/G_0 \cap L$ and associated to the principal bundle $G \to G/G_0L$. Theorem 3.1 applies to this bundle and semi-group because G/G_0L being finite, the action of G is by permutations and since S generates G , S is controllable on G/G_0L . We get thus the equality of the indices of controllability.

Observe that theorem 3.1 also relates the invariant control sets on G/L and on $G_0/L \cap G_0$: If C is an S-i.c.s. on G/L then $C \cap (G_0/G_0 \cap L)$ is an S_0-i.c.s. where S_0 is the subsemi-group of S which leaves invariant the connected component $G_0/G_0 \cap L$ of G/L . //

Now, let us see how to lift invariant control sets in the set up of example E3.3, that is, G is a Lie group, which we assume to be connected, $L_1 \subset L_2$ are closed subgroups with G/L_1 compact and S is a semi-group with non void interior in G . The fact that G/L_1 is compact implies that G/L_2 and L_2/L_1 are compact. Unless otherwise mentioned, we view $\pi_E : G/L_1 \to G/L_2$ as a fibre bundle associated to $\pi_Q : G \to G/L_2$.

Let C be an S-i.c.s. on G/L_2 and C_0 be as in proposition E3.3.1. Assume without loss of generality that the coset $L_2 \in G/L_2$ belongs to C_0 . Then by remark (i) in E3.3, $\text{int}\, S \cap L_2 \neq \emptyset$. Put $S_0 = S \cap L_2$ (this is the semi-group S_q of §2 when $\pi_Q : Q \to M$ is $\pi_Q : G \to G/L_2$ and q is the identity in G) and let $\text{gen}(S_0)$ be the subgroup of L_2 generated by S_0. Since S_0 has non void interior, $gen(S_0)$ contains the identity component L_{20} of L_2. Therefore $L_2/gen(S_0)$ is discrete, $gen(S_0)$ is closed and any orbit of $gen(S_0)$ on L_2/L_1 is a union of connected components of L_2/L_1. With these facts in mind let us state

Theorem 6.2 : Keep the situation as above.

Then

i) To have the number of S-i.c.s's over C, proceed as follows: Compute the number of S_0-i.c.s's in the $gen(S_0)$-orbits on L_2/L_1 and then sum up over the orbits.

ii) The number of S-i.c.s's over C is bounded above by the product of $ic(L_{20}/L_1 \cap L_{20})$ by the number of $gen(S_0)$-orbits.

iii) We have,

$$ic(G/L_1) \leq ic(G/L_2)ic(L_{20}/L_1 \cap L_{20})|L_2/L_1| \qquad (6.1)$$

where $|L_2/L_1|$ denotes the number of connected components of L_2/L_1.

Proof: i) Denote by Gen(S) the group of local diffeomorphisms generated by the restriction of S to $\pi_0^{-1}(C_0) \subset G$. Let us check that the orbits of $Gen(S)$ are sub-bundles of $\pi_Q^{-1}(C_0)$.

For this it is enough to see that the orbit $Gen(S)(1)$ of the identity in G is a sub-bundle. Now, $Gen(S)(1) \cap L_2$ is easily seen to be a subgroup of L_2 which contains S_0 and hence L_{20} so that it is in fact a Lie subgroup of L_2 . Let $U \subset C_0$ be a neighbourhood of $L_2 \in G/L_2$ such that $\pi_Q^{-1}(U) \tilde{\approx} U \times L_2$. Then $Gen(S)(1) \cap \pi_Q^{-1}(U) = (U \times L_{20}).\ (Gen(S)(1) \cap L_2)$ and this implies that $Gen(S)(1)$ is a submanifold of $\pi_Q^{-1}(C_0)$ which is clearly a sub-bundle.

By proposition 2.8, the structure group of $Gen(S)(1)$ is $gen(S_0)$. Fixing an orbit of $gen\ S_0$ on L_2/L_1 and constructing the corresponding bundle associated to $Gen(S)(1)$, we get the situation of proposition 3.6, which implies i).

ii) follows from i), proposition 6.1 and the fact that the identity component of $gen(S_0)$ is L_{20} .

Finally iii) is a direct consequence of ii). //

In case L_1 is normal in L_2 this theorem can be improved:

<u>Corollary 6.3</u> : Keeping the same situation as before, assume in addition that L_1 is normal in L_2 .

Then if C_1 and C_2 are two S-i.c.s's over C , there exists $g \in L_2$ depending only on L_2/ L_1 such that $g(C_1) = C_2$.

Also, (6.1) reduces to

$$ic(G/L_1) \leq ic(G/L_2)|L_2/L_1| \quad . \tag{6.2}$$

Proof: In this case $G/L_1 \to G/L_2$ is a principal bundle and as in the proof of the theorem the orbits of $Gen(S)$ are sub-bundles. Since L_2/L_1 is compact, proposition 2.9 shows that the S-i.c.s's over C are the $Gen(S)$-orbits. From this the corollary follows. //

Theorem 6.2 is useful in computing the index of controllability of compact homogeneous spaces. The idea of using it is by relating via some fibration an arbitrary G/L with other homogeneous spaces for which the index of controllability is already known. In theorem 6.6 this procedure is followed in order to get an upper bound to $ic(G/L)$. There we compare G/L with the spaces of §5 and the controllable homogeneous spaces which are going to be discussed now.

Lemma 6.4 : Let G be a Lie group and L a closed subgroup. Suppose that there exists a finite measure m on the Borel subsets of G/L , with $\operatorname{supp} m = G/L$ and which is invariant by the action of G .

Then G/L is controllable.

Proof: If $S \subset G$ is a semi-group with $\operatorname{int} S \neq \emptyset$ and which generates G , then S is controllable on G/L as follows by the arguments in lemma 1.4 and corollary 1.5. //

Proposition : Let G and L be as before with G connected and G/L compact.

Then in each of the following two cases G/L is controllable.

i) G is a compact extension of a solvable group, that is, there exists a closed solvable and normal $R \subset G$ with G/R compact.

ii) L is discrete.

Proof: In each of these cases there is a measure on G/L as in lemma 6.4.

i) For these G's the existence of an invariant probability was proved by Furstenberg [11] and Mostow [28].

ii) The assignment $g \in G \to b(g) = \det Ad(g)$ defines a continuous homomorphism from G into the positive reals (because G is connected). Since L is discrete, b is a multiplier of a semi-invariant volume ν on G/L (in the sense of [28], §2). G/L being compact, the measure induced by ν is semi-invariant and finite and thus invariant (c.f. (2.2.4), in [28]). A fortiori G is unimodular. //

In the sequel we use the following notations: If $\underset{\sim}{g}$ is the Lie algebra of G and $\underset{\sim}{h} \subset \underset{\sim}{g}$ is a subalgebra then $\underset{\sim}{n}(\underset{\sim}{h})$ denotes its normalizer in $\underset{\sim}{g}$ while $N(\underset{\sim}{h})$ is its normalizer in G. The Lie algebra of $N(\underset{\sim}{h})$ is $\underset{\sim}{n}(\underset{\sim}{h})$. Also, as in theorem 6.2, $|I/J|$ stands for the number of connected components of the homogeneous space I/J .

Theorem 6.6 : Let G be a connected Lie group and L a closed subgroup with G/L compact. Denoting by $\underset{\sim}{\ell}$ the Lie algebra of L and by $\underset{\sim}{r}$ the nilpotent radical of $\underset{\sim}{g}$, we have

a) If $\underset{\sim}{g}$ is reductive or if $\underset{\sim}{r} \subset \underset{\sim}{\ell}$ then

$$ic(G/L) \leq |N(\underset{\sim}{\ell})/L| \quad . \tag{6.3}$$

b) And in general,

$$ic(G/L) \leq |N(\underset{\sim}{\ell})/L| \; |N(\underset{\sim}{n}(\underset{\sim}{\ell}))/N(\underset{\sim}{\ell})| \quad . \tag{6.4}$$

Note: (6.3) and (6.4) are meaningful because $L \subset N(\underset{\sim}{\ell}) \subset N(\underset{\sim}{n}(\underset{\sim}{\ell}))$. Also, $N(\underset{\sim}{\ell})/L$ and $N(\underset{\sim}{n}(\underset{\sim}{\ell}))/N(\underset{\sim}{\ell})$ are compact hence (6.4) says in particular that $ic(G/L)$ is finite.

Proof: a) We apply theorem 6.2 and inequality (6.1) to the fibre bundle

$$G/L \to G/N(\underset{\sim}{\ell}) \ .$$

Let us verify that $ic(N(\underset{\sim}{\ell})_0/L \cap N(\underset{\sim}{\ell})_0) = 1$. Indeed, L_0 is normal in $N(\underset{\sim}{\ell})_0$ and is contained in $L \cap N(\underset{\sim}{\ell})_0$ so that it can be factored out and $N(\underset{\sim}{\ell})_0/L \cap N(\underset{\sim}{\ell})_0$ becomes $(N(\underset{\sim}{\ell})_0/L_0)/(L \cap N(\underset{\sim}{\ell})_0/L_0)$, which in view of proposition 6.5 ii) is controllable because $L \cap N(\underset{\sim}{\ell})_0/L_0$ is discrete.

Therefore, (6.1) for the above fibration reads

$$ic(G/L) \leq ic(G/N(\underset{\sim}{\ell}))\ |\ N(\underset{\sim}{\ell})/L\ | \ . \qquad (6.5)$$

Now, assume $\dim \underset{\sim}{\ell} = k > 0$ (otherwise $N(\underset{\sim}{\ell}) = G$ and we are in a trivial case). Then $G/N(\underset{\sim}{\ell})$ is the orbit of $\underset{\sim}{\ell}$ by the adjoint action of G on the Grassmannian of k-planes in $\underset{\sim}{g}$. Hence (6.3) for reductive $\underset{\sim}{g}$ is a consequence of theorem 5.5 (and the remark following it), the compactness of $G/N(\underset{\sim}{\ell})$ and (6.5) above.

To see (6.3) when $\underset{\sim}{r} \subset \underset{\sim}{\ell}$, assume (without loss of generality) that G is simply connected and denote by R the connected subgroup whose Lie algebra is $\underset{\sim}{r}$. Then R is normal and connected in a simply connected group, hence it is closed (c.f. [18] th. 2.1 ch. XII). Also, $R \subset L$ and if G is not solvable then G/R is reductive (i.e., $\underset{\sim}{g}/\underset{\sim}{r}$ is a reductive Lie algebra as can be seen from corollary 1 §3.9 and theorem 3.10 in [19]). Part a) is then completed by applying the reductive case to $(G/R)/(L/R)$. (If G is solvable, (6.3) is true because of proposition 6.5 i)).

Part b) is - in view of (6.5) - a consequence of the inequality

$$ic(G/N(\underset{\sim}{\ell})) \leq |N(n(\underset{\sim}{\ell}))/N(\underset{\sim}{\ell})| \ .$$

whose proof is as follows: $\underset{\sim}{r}$ is a nilpotent Lie algebra, hence by Engel's theorem its adjoint representation on $\underset{\sim}{g}$ can be put in triangular form. Therefore the group $R = \exp \underset{\sim}{r}$ can also be put in triangular form. Now, $G/N(\underset{\sim}{\ell})$ is a closed R-invariant subset of a Grassmannian, therefore R has a fixed point in $G/N(\underset{\sim}{\ell})$ (c.f. [30]).

It follows that $\underset{\sim}{r}$ is in the isotropy algebra of this point and since $\underset{\sim}{r}$ is an ideal, $\underset{\sim}{r} \subset \underset{\sim}{n}(\underset{\sim}{\ell})$. The above inequality is a consequence of a) applied to $G/N(\underset{\sim}{\ell})$. //

In case G is semi-simple and $N(\underset{\sim}{\ell})$ is parabolic, (6.3) becomes an equality:

Theorem 6.7 : Let G and L be as before and assume moreover that G is semi-simple noncompact and $N(\underset{\sim}{\ell})$ is parabolic.

Then

$$ic(G/L) = |N(\underset{\sim}{\ell})/L| \; . \qquad (6.6)$$

Proof: Suppose that $S \subset G$ is a connected semi-group with $\operatorname{int} S \neq \emptyset$ and such that if C is the unique S - i.c.s. on $G/N(\underset{\sim}{\ell})$ then $\pi^{-1}(C)$ has $|N(\underset{\sim}{\ell})/L|$ connected components in G/L , where $\pi: G/L \to G/N(\underset{\sim}{\ell})$ is the canonical fibering. Then the number of S-i.c.s's on G/L is exactly $|N(\underset{\sim}{\ell})/L|$ as is readily seen from the lifting in theorem 6.2.

Therefore, (6.6) follows if we construct a semi-group S with small enough invariant control sets on the boundaries of G . This is done in the next lemma where we check the smallness of C only on the maximum boundary, which is clearly sufficient. //

Lemma 6.8 : Let G be semi-simple connected and non compact and B its maximum boundary. Fix $b \in B$ and a neighbourhood U of b.

Then there exists a connected semi-group $S \subset G$ with $\operatorname{int} S \neq \emptyset$ and such that if C denotes the unique S-i.c.s. on B then $C \subset U$.

Proof: Fix an Iwasawa decomposition $G = KAN^+$, write $B = G/MAN^+$ (M = centralizer of A in K) and assume without loss of generality that $b = MAN^+$. We will work inside the open component $N^-MAN^+ = N^-b$ of the Bruhat decomposition of B.

Take some $H \in \underset{\sim}{a}^+$ (= the positive Weyl chamber). Let us show that H - as a vector field in B - points towards the interior of some small sphere T around b.

If $n \in N^-$ then $e^{tH} n b = e^{tH} n e^{-tH} b$, hence $\exp tH$ acts on N^-b by its adjoint in N^-. But this action is equivalent to the action of $\mathrm{Ad}(\exp tH)$ in $\underset{\sim}{n}^-$ ($N^- = \exp \underset{\sim}{n}^-$), that is, to the action of the one parameter group of the vector field on $\underset{\sim}{n}^-$ defined by means of the linear map $\mathrm{ad}(H) : \underset{\sim}{n}^- \to \underset{\sim}{n}^-$. However, the restriction of $\mathrm{ad}(H)$ to $\underset{\sim}{n}^-$ is diagonal with all eigenvalues less than zero, so that it is the gradient of a negative definite quadratic form. Any level surface of this quadratic form can then play the role of the sphere T above.

Now, given $x \in T$ there exists a neighbourhood V_x of H in $\underset{\sim}{g}$ and a neighbourhood U_x of x in T such that if $X \in V_x$ and $y \in U_x$ then $X(y)$ (X viewed as a vector field in B) points inward the sphere T. By compactness of T we can then find a neighbourhood V of H in $\underset{\sim}{g}$ such that if $X \in V$ then X is pointing towards the interior of T.

As a family of right invariant vector fields in G, V generates a semi-group which satisfies the requirements of the lemma. //

As a particular instance of the above theorem, we have the following result already proved in Oeljeklaus [29].

Corollary 6.9 : If G/L has positive Euler-Poincare characteristic, then (6.6) is satisfied.

Proof: Assume without loss of generality that G acts effectively on G/L. Then G is semi-simple and the Lie algebra $\underset{\sim}{\ell}$ of L is parabolic (c.f. [15] th. 2.4). Thus $N(\underset{\sim}{\ell})$ is parabolic and we are in the situation of theorem 6.7. //

The next corollary to theorem 6.7 relates the index of controllability of G/L with its fundamental group. We denote by $\pi(M)$ the fundamental group of M and by $|\pi(M)|$ its order.

Corollary 2.10 : If G is semi-simple noncompact, $\underset{\sim}{\ell}$ is parabolic and $|\pi(G/L)| < \infty$ then

$$ic(G/L) = \frac{|\pi G/N(\underset{\sim}{\ell}))|}{|\pi(G/L)|} \tag{6.7}$$

Proof: Assume G simply connected. Then G also acts transitively on the universal covering of G/L, which as a G-homogeneous space is given by G/L_0. (6.7) is then obtained from (6.6) by using the fibrations

$$G/L_0 \to G/L \to G/N(\underset{\sim}{\ell})$$

together with $|\pi(G/N(\underset{\sim}{\ell}))| = |N(\underset{\sim}{\ell})/L_0|$ and $|\pi(G/L)| = |L/L_0|$. //

Remark: $\pi(G/N(\underset{\sim}{\ell}))$ that appears in (6.7) is the fundamental group of a space that depends only of $\underset{\sim}{g}$ and $\underset{\sim}{\ell}$ hence is completely determined by the local action of G on $M = G/L$.

Examples:

(1) If G is semi-simple connected noncompact with finite centre and $G = KAN$ is an Iwasawa decomposition, the homogeneous space G/AN is compact and homeomorphic to K. The normalizer of the connected group AN is MAN. Therefore, by theorem 6.7 $ic(G/AN)$ is the number of connected components of MAN, which is the same as the number of components of M. This number can be computed by algebraic means (c.f. Warner [45] lemma 1.1.3.8).

(2) If in the above example, $G = Sl(n,\mathbb{R})$, $K = SO(n,\mathbb{R})$ and AN = upper triangular matrices, then G/AN is the manifold of orthonormal frames in $\mathbb{R}^n$. In this case M is a discrete group and its order is 2^{n-1} . Hence $ic(G/AN) = 2^{n-1}$.

(3) Let $St_k(n)$ be the Stiefel manifold of all orthonormal k-frames in $\mathbb{R}^n$. If $g \in Sl(n,\mathbb{R})$ and $b \in St_k(n)$, there is a well defined element $gb \in St_k(n)$ obtained by applying g to the elements of the k-frame b and orthonormalizing the result. This defines a transitive action of $Sl(n,\mathbb{R})$ on $St_k(n)$.

$St_k(n)$ fibres canonically over the Grassmannian $Gr_k(n)$ and the projection $\pi: St_k(n) \to Gr_k(n)$ is equivariant by the action of $Sl(n,\mathbb{R})$. The number of components of the typical fibre in $St_k(n) \to Gr_k(n)$ is 2 , therefore by theorem 6.7, $ic(St_k(n)\ ,\ Sl(n,\mathbb{R})) = 2$.

In particular, when $k = 1$ we have $ic(S^{n-1}\ ,\ Sl(n,\mathbb{R})) = 2$. As is shown by the next example, this is not necessarily true if $Sl(n,\mathbb{R})$ is changed by other linear group that acts transitively on S^{n-1} .

(4) Take $G = Sl(2,\mathbb{C})$ and its representation on $\mathbb{R}^4$:

$$P = A + iB \in Sl(2,\mathbb{C}) \to \begin{vmatrix} A & -B \\ B & A \end{vmatrix} \in Sl(4,\mathbb{R}) .$$

Via this representation G acts transitively on $\mathbb{R}^4 - \{0\}$ and thus on S^3 and $\mathbb{R}P^3$. An Iwasawa decomposition $G = KAN$ is given by

$K = SU(2)$, $A = \{diag(\lambda,\lambda^{-1}) : \lambda > 0\}$ and N the group of nilpotent matrices

$$N = \{\begin{vmatrix} 1 & z \\ 0 & 1 \end{vmatrix} : z \in \mathbb{C}\} \ .$$

The centralizer M of A in $SU(2)$ is the group

$$M = \{\begin{vmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{vmatrix}\} \ .$$

As homogeneous spaces, we have $S^3 = G/AN$ and $\mathbb{R}P^3 = G/DAN$ where $D = \{id, \begin{bmatrix} -1 & \\ & -1 \end{bmatrix}\} \subset M$. The maximum boundary G/MAN is the complex projective space $\mathbb{C}P^1$ and the canonical fibration

$$S^3 = G/AN \to \mathbb{C}P^1 = G/MAN$$

is the Hopf fibration.

In this case M is connected, therefore by example (1) above S^3 is unique controllable as a homogeneous space of $Sl(2,\mathbb{C})$. The proof of this fact can be rephased the following way:

S being a semi-group in $Sl(n,\mathbb{C})$, by lemma 4.2 we can assume without loss of generality that there exists $h = diag(\lambda,\lambda^{-1}) \in int\ S$ with $\lambda > 1$. After applying the representation on $\mathbb{R}^4$, h becomes

$\mathrm{diag}(\lambda,\lambda^{-1},\lambda,\lambda^{-1})$. Hence int S has a non void intersection with the subgroup of $Sl(2,\mathbb{C})$ which leaves invariant the subspace $\mathrm{span}\{e_1,e_3\} = \{(x,0,y,0) : x,y \in \mathbb{R}\}$. This subgroup being MAN , it is transitive on $\mathrm{span}\{e_1,e_3\} \cap S^3$ and since h restricted to $\mathrm{span}\{e_1,e_3\}$ is identity, one sees that int $S \cap L$ is controllable on $\mathrm{span}\{e_1,e_3\} \cap S^3$. However, $\lim_{k\to\infty} h^k v \in \mathrm{span}\{e_1,e_3\}$ for v in a dense subset of S^3 so that any S-i.c.s. on S^3 intercepts $\mathrm{span}\{e_1,e_3\} \cap S^3$ and S has at most one invariant control set.

The equivariant action of $Sl(2,\mathbb{C})$ on the fibre bundle $S^3 \to \mathbb{R}P^3$ provides an example in which the inequality in (6.1) is strict.

In fact, the left hand side of (6.1) is $ic(S^3,Sl(2,\mathbb{C}))$ which is 1, whereas the right hand side equals to $DAN/AN = 2$.

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